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MATHEMATICAL MODELLING OF MULTI CONDUCTOR CABLES

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Abstract. This paper proposes a formal justification of simplified 1D models for the propagation of electromagnetic waves in thin non-homogeneous lossy conductor cables. Our approach consists in deriving these models from an asymptotic analysis of 3D Maxwell's equations. In essence, we extend and complete previous results to the multi-wires case.

1. **Introduction.** The present paper is the continuation of the article [9]. We aim at proposing a rigorous justification of simplified 1D models for the propagation of electromagnetic waves in thin conductor cables (also called transmission lines in the literature [12]). Our approach consists in deriving such models from an asymptotic analysis of 3D Maxwell's equations, considering the transverse dimensions of the cable as the small parameter in the analysis. In this sense, our approach is similar to what has been done in mechanics for deriving the theory of beams from the equations of 3D elasticity [1], [14]. It differs by the domain of application (electromagnetism) and the fact that we are considering evolution problems. Doing so, we justify and extend the telegrapher's models classically used by electrical engineers [12], [15], [16].

In [9], we treated the case of a co-axial cable containing a single metallic wire. In the present paper, we essentially extend and complete the results of [9] to the multi-wires case. Note that, as in [9], we shall restrict ourselves to the formal derivation of the model. The corresponding error analysis is postponed to a future work. Note however that a first step has been done in this direction in [10].

We must emphasize that the situation we consider in this paper is rather general: the cable has a variable cross section of arbitrary geometry and the dielectric medium

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in which the electromagnetic field is propagating is heterogeneous and lossy. This degree of generality is mandatory in the application context that we have in mind, namely the non destructive testing of electric networks: the presence of defects or localized damages modifies the nice structure of a perfect cable. From the mathematical point of view, the absence of any cylindrical structure prevents us from using some tools such has Fourier transform in space or modal decomposition (separation of variables), which is done in many textbooks [6], [12], [13].

The outline of this article is as follows. Section 2 is devoted to preliminary material about 2D vector fields. The problem under consideration is presented in detail in section 3. In section 4, we perform the formal asymptotic analysis of this problem and establish, using the results of section 2, our generalized telegrapher's model (90). Finally, in section 5, various properties of this model are established.

We intend this paper to put the theory of transmission lines into a fresh perspective that will be of interest to both physicists and mathematicians. Future work will be devoted to completing the mathematical analysis of our models (error estimates) and developing numerical methods for their resolution, as well as the corresponding numerical analysis.

2. **Spaces of quasi-static vector fields and their properties.**

2.1. **Preliminaries and notation.** In this section we establish or recall useful results about 2D vector fields. In particular, these results extend, in another context, those of [3], chapters 7 and 9. In what follows, all manifolds that will be introduced will be supposed at least Lipschitz continuous. The proof of some results sometimes requires additional regularity that will not be explicitly mentioned. We are however convinced, that our results are still valid after removing these regularity assumptions.

We define a 2D domain S in the plane $x = (x_1, x_2)$, with canonical basis $(\mathbf{e}_1, \mathbf{e}_2)$, that has the following structure

$$S = \mathcal{O} \setminus \overline{\bigcup_{j}^{N} \mathcal{O}_j}, \quad \overline{\mathcal{O}_k} \cap \overline{\mathcal{O}_\ell} \neq \emptyset \text{ for } k \neq \ell, \tag{1}$$

where $\mathcal{O}$ is a bounded simply connected open set and the $\mathcal{O}_j$'s are simply connected open subsets of $\mathcal{O}$ (holes) whose closures do not intersect each other. As a consequence, the boundary of S is the union of an exterior boundary and N interior boundaries, each of them being closed curves:

$$\partial S = \partial S_e \bigcup_{j=1}^{N} \Sigma_j \quad \text{with} \quad \partial S_e = \partial \mathcal{O} \quad \text{and} \quad \Sigma_j = \partial \mathcal{O}_j. \tag{2}$$

Along ∂S we shall denote $n = (n_1, n_2)$ the unit normal vector which is outgoing with respect to S. Finally, we assume that there exists N cuts Γ_j such that

(i) each Γ_j is a (possibly curved) line joining Σ_j to ∂S_e,

(ii) denoting $\Gamma = \displaystyle\bigcup_{j=1}^{N} \Gamma_j$, $S_\Gamma := S \setminus \Gamma$ is simply connected

In the sequel, we shall say that:

$$\text{A 2D domain } S \text{ satisfying (1), (2), (i) and (ii) belongs to the class } \mathcal{C}_N. \tag{3}$$

In what follows, each Γ_j will be parameterized by a curvilinear abscissa $\tau \in [0, L_j]$ (L_j is the length of Γ_j) that orients (when τ increases) Γ_j from ∂S_e towards Σ_j. Denoting also τ the corresponding unit tangent vector along Γ_j, we shall define n along Γ_j as the unit normal vector to Γ_j such that (τ, n) is a direct basis of the plane. All these assumptions and notation are summarized by figure 1. In the

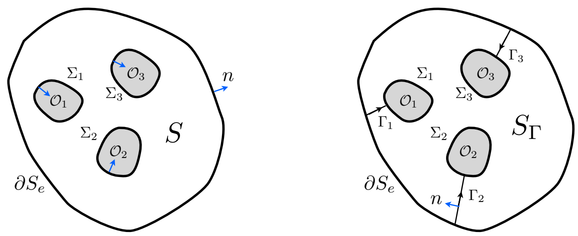


FIGURE 1. The geometry of the domain S.

sequel 2D vector fields will be implicitly identified to 3D vector fields whose third component vanish and scalar fields to 3D vector fields which are normal to the plane (x_1, x_2). Therefore, for any $\mathbf{v} = (v_1, v_2) \in \mathbb{C}^2$, one can write

$$n \times \mathbf{v} = -\,\mathbf{v} \times n = n_1\, v_2 - n_2\, v_1, \quad \mathbf{e}_3 \times \mathbf{v} = -\,\mathbf{v} \times \mathbf{e}_3 = (-v_2, v_1), \tag{4}$$

so that

$$(\mathbf{e}_3 \times \mathbf{v}) \cdot n = \mathbf{v} \times n. \tag{5}$$

The divergence and scalar rotational of a 2D vector field $\mathbf{v} = (v_1, v_2)$ are defined by (using an obvious notation for partial derivatives)

$$\operatorname{div} \mathbf{v} = \partial_1 v_1 + \partial_2 v_2, \quad \operatorname{rot} \mathbf{v} = \partial_1 v_2 - \partial_2 v_1. \tag{6}$$

and one has the obvious relationships

$$\operatorname{div} \mathbf{v} = \operatorname{rot}(\mathbf{e}_3 \times \mathbf{v}), \quad \operatorname{rot} \mathbf{v} = \operatorname{div}(\mathbf{v} \times \mathbf{e}_3). \tag{7}$$

Analogously, one defines the gradient and vector rotational of a scalar field φ as

$$\nabla\varphi = (\partial_1\varphi, \partial_2\varphi), \quad \mathbf{rot}\, \varphi = (\partial_2\varphi, -\partial_1\varphi), \tag{8}$$

in such a way that

$$\nabla\varphi = -\,\mathbf{rot}\,\varphi \times \mathbf{e_3}, \quad \mathbf{rot}\,\varphi = -\,\mathbf{e_3} \times \nabla\varphi, \tag{9}$$

as well as

$$\operatorname{rot}(\nabla\varphi) = 0, \quad \operatorname{div}(\mathbf{rot}\,\varphi) = \mathbf{0}. \tag{10}$$

Given $\varphi \in C^0\big(\overline{S_\Gamma}\big)$, we shall denote the jump of φ, denoted $[\varphi]$, along Γ as:

$$\forall\, x \in \Gamma_j, \quad [\varphi](x) := \lim_{\eta \searrow 0} \big[\, u(x + \eta\, n) - u(x - \eta\, n) \,\big] \tag{11}$$

We shall use the following Green's formula, valid for $(\varphi, \mathbf{v}) \in C^1\big(\overline{S_\Gamma}\big) \times C^1\big(\overline{S_\Gamma}\big)^2$:

$$\int_{S_\Gamma} \operatorname{div} \mathbf{v}\, \varphi = -\int_{S_\Gamma} \mathbf{v} \cdot \nabla\varphi + \int_{\partial S} \mathbf{v} \cdot n\, \varphi + \int_\Gamma [\varphi\, \mathbf{v}] \cdot n \tag{12}$$

$$\int_{S_\Gamma} \operatorname{rot} \mathbf{v}\, \varphi = \int_{S_\Gamma} \mathbf{v} \cdot \mathbf{rot}\,\varphi - \int_{\partial S} \mathbf{v} \times n\, \varphi - \int_\Gamma [\varphi\, \mathbf{v}] \times n \tag{13}$$

where $\int_\Gamma$ stands for $\sum_{j=1}^{N}\int_{\Gamma_j}$ and $\int_{\partial S}$ stands for $\int_{\partial S_e}+\sum_{j=1}^{N}\int_{\Sigma_j}$.

Remark 1. *Both formulas (12, 13) are in fact deduced the one from the other thanks to (9). Moreover, it is well known that formulas (12, 13) can appropriately be extended to $H(div, S)\times H^1(\Gamma)$ and $H(rot, S)\times H^1(\Gamma)$ respectively, integrals over Γ being changed into duality brackets.*

2.2. **Quasi-static vector fields.** We shall say that $\rho(x) : S \mapsto \mathbb{C}$ is in the class $\mathcal{N}(S)$ if it is bounded and its real part is strictly positive:

$$\text{a. e. } x\in S,\quad 0<\rho_-\le \mathcal{R}e\,\rho(x). \tag{14}$$

Note that in $\rho\in\mathcal{N}(S)$, $\rho^{-1}\,(\equiv 1/\rho\,)\in\mathcal{N}(S)$. Denoting

$$\mathcal{V}(S)=\big\{\,\mathbf{v}\in L^2(S)^2\,/\,\mathrm{rot}\,\mathbf{v}=0\big\}, \tag{15}$$

we introduce the two spaces of quasi-static vector fields respectively of electric and magnetic type (which are known respectively as the harmonic Dirichlet and Neumann fields when ρ is constant):

$$\mathcal{E}(\rho;S)=\big\{\mathbf{v}\in\mathcal{V}(S)\,/\,\mathrm{div}(\rho\,\mathbf{v})=0,\mathbf{v}\times n=0\text{ on }\partial S\big\}, \tag{16}$$

$$\mathcal{H}(\rho;S)=\big\{\mathbf{v}\in\mathcal{V}(S)\,/\,\mathrm{div}(\rho\,\mathbf{v})=0,\mathbf{v}\cdot n=0\text{ on }\partial S\big\}. \tag{17}$$

By adapting, to the dimension 2 and to the case where the function ρ is not constant, the proof of the propositions 3.14 and 3.18 of [2] (see also [8], section 3.1, theorem 3.1 for an analogous result), one shows that $\mathcal{E}(\rho;S)$ and $\mathcal{H}(\rho;S)$ are N-dimensional spaces and more precisely that

$$\mathcal{E}(\rho;S)=\mathrm{span}\big\{\nabla\varphi_j(\rho,S),\ 1\le j\le N\big\},\ \ \mathcal{H}(\rho;S)=\mathrm{span}\big\{\widetilde{\nabla}\psi_j(\rho,S),\ 1\le j\le N\big\},$$

where the notation $\widetilde{\nabla}$ is defined in Remark 2 and the $\varphi_j(\rho,S)\in H^1(S)$ are the solutions of the boundary value problems:

$$\left\{\begin{array}{ll}\mathrm{div}\big(\rho\,\nabla\varphi_j(\rho,S)\big)=0, & \text{in } S,\\[2mm] \varphi_j(\rho,S)=0, & \text{on }\partial S^e,\\[2mm] \varphi_j(\rho,S)=\delta_{jk}, & \text{on }\Sigma_k,\ k\le N,\end{array}\right. \tag{18}$$

while the $\psi_j(\rho,S)\in H^1(S_\Gamma)$ are the solutions of

$$\left\{\begin{array}{ll}\mathrm{div}\big(\rho\,\nabla\psi_j(\rho,S)\big)=0, & \text{in } S_\Gamma,\\[2mm] \nabla\psi_j(\rho,S)\cdot n=0, & \text{on }\partial S,\\[2mm] \big[\psi_j(\rho,S)\big]=\delta_{jk},\quad \big[\rho\,\partial_n\psi_j(\rho,S)\big]=0 & \text{on }\Gamma_k,\ k\le N.\end{array}\right. \tag{19}$$

The well-posedness of (18) and (19) follows from Lax-Milgram's lemma. The solution of (19) is unique only up to an additive constant (its gradient is thus unique).

Remark 2. *For any $\psi\in H^1(S_\Gamma)$, the field $\widetilde{\nabla}\psi$ is defined by*

$$\widetilde{\nabla}\psi\in L^2(S)^2\quad\textit{and}\quad\widetilde{\nabla}\psi=\nabla\psi\quad\textit{in }\mathcal{D}'(S_\Gamma).$$

One defines analogously $\widetilde{\mathbf{rot}}\,\psi$.

Note that if $\psi_j(\rho, S)$ does depend on the position of the cuts Γ_k, $\widetilde{\nabla}\psi_j(\rho, S)$ does not. This is not so obvious but will become clear later thanks to Lemma 2.4, (27).

Remark 3. *Referring to the weak formulation of (18) and (19) respectively, one sees that:*

$$\begin{cases} \displaystyle\int_S \rho\, \nabla\varphi_j(\rho, S) \cdot \nabla\varphi^* = 0, & \forall\, \varphi^* \in H_0^1(S), \\ \displaystyle\int_S \rho \widetilde{\nabla}\, \psi_j(\rho, S) \cdot \nabla\psi^* = 0, & \forall\, \psi^* \in H^1(S). \end{cases} \tag{20}$$

It is easy to check that the fields $\nabla\varphi_j(\rho, S)$ on one hand, and similarly $\widetilde{\nabla}\psi_j(\rho, S)$ on the other hand, are linearly independent:

- If $\sum\ \alpha_j \nabla\varphi_j(\rho, S) = 0$, by connectedness $\sum \alpha_j\, \varphi_j(\rho, S)$ is constant in S and this constant is 0 since each $\varphi_j(\rho, S)$ vanishes on ∂S_e. Then writing $\sum\ \alpha_j\, \varphi_j(\rho, S) = 0$ on Σ_k yields $\alpha_k = 0$.
- If $\sum\ \alpha_j \widetilde{\nabla}\psi_j(\rho, S) = 0$, by connectedness $\sum \alpha_j\, \psi_j(\rho, S)$ is constant in S. In particular $\sum\ \alpha_j\, [\psi_j(\rho, S)] = 0$ on Γ_k, which yields $\alpha_k = 0$.

Moreover, these bases satisfy particular generalized bi-orthogonality relations:

Lemma 2.1. *Let $(\rho_1, \rho_2) \in \mathcal{N}(S)^2$, for any $(k, \ell) \in \{1, ..., N\}^2$,*

$$\int_S \nabla\varphi_\ell(\rho_1, S) \cdot \overline{\widetilde{\mathbf{rot}}\, \psi_k(\rho_2, S)} \equiv -\int_S \mathbf{rot}\varphi_\ell(\rho_1, S) \cdot \overline{\widetilde{\nabla}\psi_k(\rho_2, S)} = \delta_{k\ell}. \tag{21}$$

Proof. We drop the argument S for the sake of simplicity. Let us apply Green's formula (13) with $\varphi = \psi_k(\rho_2)$ and $\mathbf{v} = \nabla\varphi_\ell(\rho_1)$. Using (10) and the fact that, since $\varphi_\ell(\rho_1)$ is constant along ∂S_e and each Σ_j, $\mathbf{v} \times n$ vanishes on ∂S, we get

$$\begin{aligned} \int_S \nabla\varphi_\ell(\rho_1) \cdot \overline{\widetilde{\mathbf{rot}}\, \psi_k(\rho_2)} &\equiv \int_{S_\Gamma} \nabla\varphi_\ell(\rho_1) \cdot \overline{\mathbf{rot}\, \psi_k(\rho_2)} \\ &= -\sum_{m=1}^{N} \int_{\Gamma_m} [\overline{\psi_k(\rho_2)}]\, \nabla\varphi_\ell(\rho_1) \times n = -\int_{\Gamma_k} \nabla\varphi_\ell(\rho_1) \times n. \end{aligned}$$

We conclude after noticing that $-\displaystyle\int_{\Gamma_k} \nabla\varphi_\ell(\rho_1) \times n = \varphi_\ell(\rho_1)|_{\Gamma_k} - \varphi_\ell(\rho_1)|_{\partial S_e}$. □

We introduce the "generalized Gramm $N \times N$ matrices"

$$\mathbf{Y}(\rho, S) = \big(\mathbf{Y}_{ij}(\rho, S)\big)_{ij}, \quad \mathbf{Y}_{ij}(\rho, S) = \int_S \rho\, \nabla\varphi_i(\rho, S) \cdot \overline{\nabla\varphi_j(\rho, S)}, \tag{22}$$

$$\mathbf{Z}(\rho, S) = \big(\mathbf{Z}_{ij}(\rho, S)\big)_{ij}, \quad \mathbf{Z}_{ij}(\rho, S) = \int_S \rho\, \widetilde{\nabla}\psi_i(\rho, S) \cdot \overline{\widetilde{\nabla}\psi_j(\rho, S)}. \tag{23}$$

Lemma 2.2. *The matrices $\mathbf{Y}(\rho, S)$ and $\mathbf{Z}(\rho, S)$ are invertible and symmetric. They are real symmetric positive definite if ρ is real.*

Proof. We give the proof for $\mathbf{Y}(\rho, S)$. The proof for $\mathbf{Z}(\rho, S)$ is similar.

To show the symmetry, we apply (20) with $\varphi^* = \varphi_j(\rho, S) - \overline{\varphi_j(\rho, S)}$ to observe that

$$\mathbf{Y}_{ij}(\rho, S) = \int_S \rho\, \nabla\varphi_i(\rho, S) \cdot \nabla\varphi_j(\rho, S). \tag{24}$$

For the invertibility, let $u \in \mathbb{C}^N$ and $\varphi_\rho(u) := \sum_{j=1}^{N} u_j \, \varphi_j(\rho, S)$

$$\forall\,(u,v) \in \mathbb{C}^N \times \mathbb{C}^N, \quad \big(\mathbf{Z}(\rho,S)\,u,v\big)_{\mathbb{C}^N} = \int_S \rho\,\nabla\varphi_\rho(u)\cdot\overline{\nabla\varphi_\rho(v)}. \tag{25}$$

In particular

$$\Big(\mathbf{Z}(\rho,S)\,u,u\Big)_{\mathbb{C}^N} = 0 \implies \int_S (\mathcal{R}e\,\rho)\,|\nabla\varphi_\rho(u)|^2 = 0 \implies \varphi_\rho(u) = 0 \implies u = 0.$$

This also shows that when $\mathcal{I}m\,\rho = 0$, $\mathbf{Z}(\rho)$ is real symmetric positive definite. □

Let us consider the linear map in $L^2(S)^2$

$$J_\rho : \mathbf{v} \quad \longrightarrow \quad \rho\,\mathbf{e}_3 \times \mathbf{v} \tag{26}$$

which is an isomorphism with inverse $J_\rho^{-1} = -\,J_{1/\rho}$.

Lemma 2.3. *J_ρ is an isomorphism from $\mathcal{E}(\rho\,;S)$ into $\mathcal{H}(\rho^{-1};S)$ and from $\mathcal{H}(\rho\,;S)$ into $\mathcal{E}(\rho^{-1};S)$.*

Proof. If $\mathbf{v} \in \mathcal{E}(\rho;S)$ then, using (7)

$$\operatorname{div}\big(\rho^{-1}\,J_\rho\mathbf{v}\big) = \operatorname{div}(\mathbf{e}_3\times\mathbf{v}) = -\operatorname{rot}(\mathbf{v}) = 0,$$

$$\operatorname{rot} J_\rho\mathbf{v} = \operatorname{rot}(\mathbf{e}_3\times\rho\mathbf{v}) = \operatorname{div}(\rho\mathbf{v}) = 0,$$

$$J_\rho\mathbf{v}\cdot n = \rho\,(\mathbf{e}_3\times\,\mathbf{v})\cdot n = \rho\,\mathbf{v}\times n = 0.$$

which means that $J_\rho\mathbf{v} \in \mathcal{H}(\rho^{-1};S)$. Since the two spaces have dimension N, J_ρ is an isomorphism. A similar argument stands for the second statement of the lemma. □

Lemma 2.4. *One has the formulas*

$$\widetilde{\nabla}\psi_j(\rho,S) = -\,\rho^{-1}\sum_{\ell=1}^{N} \mathbf{Z}_{j\ell}(\rho,S)\,\mathbf{rot}\,\varphi_\ell(\rho^{-1},S), \tag{27}$$

$$\nabla\varphi_j(\rho,S) = \rho^{-1}\sum_{\ell=1}^{N} \mathbf{Y}_{j\ell}(\rho,S)\,\widetilde{\mathbf{rot}}\,\psi_\ell(\rho^{-1},S), \tag{28}$$

and moreover $\mathbf{Y}(\rho,S)^{-1} = \mathbf{Z}(\rho^{-1},S)$, and thus $\mathbf{Z}(\rho,S)^{-1} = \mathbf{Y}(\rho^{-1},S)$.

Proof. For simplicity, we note $\mathcal{E}(\rho)$ for $\mathcal{E}(\rho;S)$, $\varphi_j(\rho)$ for $\varphi_j(\rho,S)$ and so on.

According to lemma 2.3, $J_\rho\widetilde{\nabla}\psi_j(\rho) \in \mathcal{E}(\rho^{-1})$ which implies, using (9)

$$\rho\,\widetilde{\nabla}\psi_j(\rho) = \sum_{\ell=1}^{N} \alpha_{j\ell}\,\mathbf{rot}\,\varphi_\ell(\rho^{-1}).$$

Therefore $\mathbf{Z}_{jk}(\rho) = \int_S \rho\,\widetilde{\nabla}\psi_j(\rho)\cdot\overline{\widetilde{\nabla}\psi_k(\rho)} = \sum_{\ell=1}^{N}\alpha_{j\ell}\int_S \mathbf{rot}\,\varphi_\ell(\rho^{-1})\cdot\overline{\widetilde{\nabla}\psi_k(\rho)}.$

Using (21) with $\rho_1 = \rho^{-1}$ and $\rho_2 = \rho$ shows that $\alpha_{jk} = -\mathbf{Z}_{jk}(\rho)$ and thus (27). (28) is shown analogously.

Taking $\rho^{-1}\,\mathbf{e}_3\times$ (27), once (27) has been written for ρ^{-1} instead of ρ, leads to

$$\rho^{-1}\,\widetilde{\mathbf{rot}}\,\psi_\ell(\rho^{-1}) = \sum_{k=1}^{N}\,\mathbf{Z}_{\ell k}(\rho^{-1})\,\nabla\varphi_k(\rho). \tag{29}$$

Substituting (29) into (28) leads to

$$\nabla\varphi_j(\rho) = \sum_{k=1}^{N}\Big(\sum_{\ell=1}^{N}\,\mathbf{Y}_{j\ell}(\rho)\,\mathbf{Z}_{\ell k}(\rho^{-1})\Big)\,\nabla\varphi_k(\rho). \tag{30}$$

By linear independence of the $\nabla\varphi_j(\rho)$, this shows $\mathbf{Y}(\rho)\,\mathbf{Z}(\rho^{-1}) = \mathbf{I}d$. □

Our next properties concern some invariance properties of the matrices $\mathbf{Y}(\rho, S)$ and $\mathbf{Z}(\rho, S)$. In the sequel $\mathcal{T}_c$ denotes a conformal mapping, defined here as a diffeomorphism of $\mathbb{R}^2$ whose Jacobian matrix $D\mathcal{T}_c(x)$ is proportional to a unitary matrix at each point:

$$\mathcal{T}_c \in C^1(\mathbb{R}^2,\mathbb{R}^2)\ /\ \forall\, x\in\mathbb{R}^2,\quad D\mathcal{T}_c(x)^t\,D\mathcal{T}_c(x) = \alpha^2(x)\,\mathbf{I}d. \tag{31}$$

Note that conformal mappings include linear transformations such as rotations, symmetries, homotheties and products of them.

Given $\rho\in\mathcal{N}(S)$, one defines $\rho_{\mathcal{T}_c}\in\mathcal{N}(S_{\mathcal{T}_c})$, where $S_{\mathcal{T}_c} := \mathcal{T}_c(S)$ as

$$\rho_{\mathcal{T}_c} = \rho\circ\mathcal{T}_c^{-1}. \tag{32}$$

Lemma 2.5. *For any conformal mapping $\mathcal{T}$, one has*

$$\mathbf{Y}(\rho_{\mathcal{T}_c}, S_{\mathcal{T}_c}) = \mathbf{Y}(\rho, S),\quad \mathbf{Z}(\rho_{\mathcal{T}_c}, S_{\mathcal{T}_c}) = \mathbf{Z}(\rho, S). \tag{33}$$

Proof. Let us give the proof of the first equality, the second being deduced by lemma 2.4. From $\varphi_j(\rho, S)\in H^1(S)$ solution of (18), we define $\varphi_{\mathcal{T}_c,j}(\rho, S)\in H^1(S_{\mathcal{T}_c})$ by

$$\varphi_{j,\mathcal{T}_c}(\rho, S) = \varphi_j(\rho, S)\circ\mathcal{T}_c^{-1}.$$

Using (31), one sees that $\varphi_{j,\mathcal{T}_c}(\rho, S)$ solves (the details are left to the reader)

$$\begin{cases} \operatorname{div}\big(\rho_{\mathcal{T}_c}\,\nabla\varphi_{j,\mathcal{T}_c}(\rho, S)\big) = 0, & \text{in } S_{\mathcal{T}_c},\\[1ex] \varphi_{j,\mathcal{T}_c}(\rho, S) = 0, & \text{on } \partial S^e_{\mathcal{T}_c} := \mathcal{T}_c(\partial S_e),\\[1ex] \varphi_{j,\mathcal{T}_c}(\rho, S) = \delta_{jk}, & \text{on } \Sigma_{k,\mathcal{T}_c} := \mathcal{T}_c(\Sigma_k),\ k\le N, \end{cases}$$

which means that $\varphi_{j,\mathcal{T}_c}(\rho, S) = \varphi_j(\rho_{\mathcal{T}_c}, S_{\mathcal{T}_c})$ namely

$$y = \mathcal{T}_c(x)\ \Rightarrow\ \big[\varphi_j(\rho_{\mathcal{T}_c}, S_{\mathcal{T}_c})\big](y) = \big[\varphi_j(\rho, S)\big](x).$$

This yields

$$y = \mathcal{T}_c(x)\quad\Rightarrow\quad \big[\nabla\varphi_j(\rho_{\mathcal{T}_c}, S_{\mathcal{T}_c})\big](y) = D\mathcal{T}_c(x)^{-1}\big[\nabla\varphi_j(\rho, S)\big](x).$$

Therefore, using the change of variable $y = \mathcal{T}_c(x)$, with Jacobian $\alpha^2(x)$ because we are in $\mathbb{R}^2$, we have

$$\begin{aligned} \mathbf{Y}_{ij}(\rho_{\mathcal{T}_c}, S_{\mathcal{T}_c}) &= \int_{S_{\mathcal{T}_c}} \rho_{\mathcal{T}}\,\nabla\varphi_i(\rho_{\mathcal{T}_c}, S_{\mathcal{T}_c})\cdot\overline{\nabla\varphi_j(\rho_{\mathcal{T}_c}, S_{\mathcal{T}_c})}\,dy\\ &= \int_{S_{\mathcal{T}_c}} \rho\,\big(D\mathcal{T}_c^t\,D\mathcal{T}_c\big)^{-1}\,\nabla\varphi_i(\rho, S)\cdot\overline{\nabla\varphi_j(\rho, S)}\ \alpha^2\,dx, \end{aligned}$$

which yields $\mathbf{Y}_{ij}(\rho_{\mathcal{T}_c}, S_{\mathcal{T}_c}) = \mathbf{Y}_{ij}(\rho, S)$ since $\alpha^2\big[D\mathcal{T}_c^t\,D\mathcal{T}_c\big]^{-1} = \mathbf{I}d$. □

Finally, let us give a property of the matrices $\mathbf{Y}(\rho, S)$ and $\mathbf{Z}(\rho, S)$ when ρ is real.

Lemma 2.6. *If ρ is real valued, then the matrix $\mathbf{Y}(\rho, S)$ is a strict M-matrix, more precisely*

$$\mathbf{Y}_{ij}(\rho, S) < 0 \;\text{ for } i \neq j, \quad \mathbf{Y}_{ii}(\rho, S) > \sum_{j \neq i} |\mathbf{Y}_{ij}(\rho, S)|, \quad \forall\, 1 \leq i \leq N, \tag{34}$$

and the matrix $\mathbf{Z}(\rho, S)$ satisfies:

$$\mathbf{Z}_{ij}(\rho, S) > 0, \quad \forall\, 1 \leq i, j \leq N. \tag{35}$$

Proof. (35) is obtained from (34), lemma 2.4 (last line) and well-known properties of inverses of M-matrices. (34) appears as a consequence of the maximum principle applied to (18), which implies that $0 \leq \varphi_j(S, \rho) \leq 1$. In particular, $\varphi_j(S, \rho)$ reaches its minimum on $\overline{S}$ along each Σ_i, for $i \neq j$. Then, $\partial_n \varphi_j(S, \rho) \leq 0$ along Σ_i. Using Green's formula we deduce

$$\mathbf{Y}_{ij}(\rho, S) = \int_{\Sigma_i} \rho\, \partial_n \varphi_j(S, \rho) \leq 0. \tag{36}$$

In fact, the above inequality is strict since $\mathbf{Y}_{ij}(\rho, S) = 0$ would imply $\partial_n \varphi_j(S, \rho) = 0$ along Σ_i. As $\varphi_j(S, \rho) = 0$ along Σ_i, by unique continuation, one would deduce that $\varphi_j(S, \rho)$ vanishes on S which contradicts the fact that $\varphi_j(S, \rho) = 1$ along Σ_j.

Then, the second inequality of (36) amounts to $\displaystyle\sum_j \mathbf{Y}_{ij}(\rho, S) > 0$. We have

$$\sum_j \mathbf{Y}_{ij}(\rho, S) = \int_S \rho\, \nabla \varphi_i(S, \rho) \cdot \nabla \varphi_e(S, \rho) \quad \text{where } \varphi_e(S, \rho) := \sum_j \varphi_j(S, \rho)$$

satisfies the boundary value problem

$$\left\{ \begin{array}{ll} \operatorname{div}\big(\rho\, \nabla \varphi_e(\rho, S)\big) = 0, & \text{in } S, \\[2mm] \varphi_e(\rho, S) = 0, & \text{on } \partial S^e, \\[2mm] \varphi_e(\rho, S) = 1, & \text{on } \Sigma_k,\ k \leq N. \end{array} \right. \tag{37}$$

Again, by the maximum principle, $\varphi_e(\rho, S)$ is maximal on each Σ_k, thus $\partial_n \varphi_e(S, \rho) \geq 0$ on Σ_i, while Green's formula leads to the identity

$$\sum_j \mathbf{Y}_{ij}(\rho, S) = \int_{\Sigma_i} \rho\, \partial_n \varphi_e(S, \rho) \geq 0.$$

The proof that this inequality is strict is similar to the proof that inequality (36) is strict. $\square$

3. **A mathematical model for multi-coaxial cables.** A cable, with axis x_3, will be defined as the union of its cross sections (in the (x_1, x_2) plane)

$$\Omega = \bigcup_{x_3 \in \mathbb{R}} S_{x_3} \times \{x_3\}, \quad S_{x_3} \in \mathcal{C}_N \quad \big(\text{cf. } (3)\big), \quad \text{for each } x_3. \tag{38}$$

We assume that, in addition, $\partial\Omega$ is a Lipschitz continuous manifold such that

$$\partial\Omega = \bigcup_{x_3 \in \mathbb{R}} \partial S_{x_3} \times \{x_3\}.$$

Along $\partial\Omega$, we define the 2D vector field of unit normal vectors $n : \partial\Omega \to \mathbb{R}^2$ such that, along ∂S_{x_3}, n is the unit normal vector to ∂S_{x_3}, outgoing with respect to S_{x_3}

(this is coherent with the notation of section 2). On $\partial\Omega$, a field of (non unitary) normal vectors to $\partial\Omega$, outgoing with respect to Ω, will be defined by:

$$\mathbf{n} := (n, g) : \partial\Omega \longrightarrow \mathbb{R}^3, \quad n : \partial\Omega \longrightarrow \mathbb{R}^2, \quad g : \partial\Omega \longrightarrow \mathbb{R}. \tag{39}$$

In other words g denotes the longitudinal component of the normal vector to $\partial\Omega$ whose projection on the (x_1, x_2) plane is n. According to the notation of section 2,

$$S_{x_3} = \mathcal{O}^{x_3} \setminus \overline{\bigcap_{j=1}^{N} \mathcal{O}_j^{x_3}},$$

where the holes $\mathcal{O}_j^{x_3}$ in S_{x_3}, with boundaries $\Sigma_j^{x_3}$, define the metallic wires (filled by perfectly conducting metal)

$$\mathcal{W}_j = \bigcup_{x_3 \in \mathbb{R}} \mathcal{O}_j^{x_3} \times \{x_3\}. \tag{40}$$

All this notation is summarized in figure 2 in the case $N = 3$.

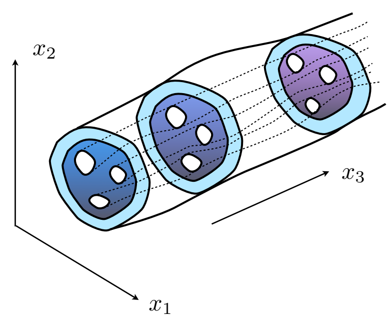


FIGURE 2. The geometry of the domain Ω.

Example : *A fundamental example is the case where the cross sections are obtained by smooth deformations of a reference section, the domain S of section 2. More precisely, let $\{\mathcal{T}_{x_3}, x_3 \in \mathbb{R}\}$ be a family of diffeomorphisms of $\mathbb{R}^2$ from which we define $\mathcal{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that*

$$\mathcal{T}(x, x_3) := \mathcal{T}_{x_3}(x), \tag{41}$$

we define the cross sections as

$$\forall\, x_3 \in \mathbb{R}, \quad S_{x_3} = \mathcal{T}_{x_3} S, \tag{42}$$

so that S_{x_3} has the same topological properties and regularity as S (see [7]). If we assume that $\mathcal{T} \in C^1(\mathbb{R}^3, \mathbb{R}^3)$, we can define the vector field $\mathcal{V} \in C^0(\mathbb{R}^3, \mathbb{R}^3)$ of Eulerian velocities (we use this vocabulary by analogy with fluid mechanics, assuming that the variable x_3 plays the role of the time), such that

$$y = \mathcal{T}(x, x_3) \equiv \mathcal{T}_{x_3}(x) \quad \Longrightarrow \quad \mathcal{V}(y, x_3) = \partial_3 \mathcal{T}(x, x_3). \tag{43}$$

It is then an exercise in differential geometry to check that $\partial\Omega$ is a Lipschitz manifold and that (39) is satisfied with

$$g = -\,\mathcal{V} \cdot n. \tag{44}$$

We shall consider a lossy dielectric material whose characteristics coefficients, electric permittivity, magnetic permeability, electric and magnetic conductivities (responsible for losses), satisfy the usual assumptions:

$$\begin{array}{lll} 0 < \varepsilon_- \le \varepsilon(\mathbf{x}) \le \varepsilon_+, & 0 < \mu_- \le \mu(\mathbf{x}) \le \mu_+ \ , & \mathbf{x} \in \Omega, \\ 0 \le \sigma_{e,-} \le \sigma_e(\mathbf{x}) \le \sigma_{e,+}, & 0 \le \sigma_{m,-} \le \sigma_m(\mathbf{x}) \le \sigma_{m,+} \ , & \mathbf{x} \in \Omega. \end{array} \tag{45}$$

We consider a family of (thin) domains parametrized by a small but strictly positive scalar δ (the diameter of the cross section in the plane $x_3 = 0$)

$$\Omega^\delta = \mathcal{G}_\delta(\Omega),$$

where $\mathcal{G}_\delta$ is the scaling transformation $\mathcal{G}_\delta : (x_1, x_2, x_3) \longrightarrow (\delta x_1, \delta x_2, x_3)$. We assume that the material properties of the thin cables are defined from the ones of the reference cable Ω according to this scaling:

$$\varepsilon^\delta = \varepsilon \circ \mathcal{G}_\delta^{-1}, \quad \mu^\delta = \mu \circ \mathcal{G}_\delta^{-1}, \quad \sigma_e^\delta = \sigma_e \circ \mathcal{G}_\delta^{-1}, \quad \sigma_m^\delta = \sigma_m \circ \mathcal{G}_\delta^{-1}.$$

Remark 4. Abuse of notation. *In what follows, we use the passage from Ω to Ω^δ via the change of variable $\mathcal{G}_\delta$. To make thing clearer, we use the same letters $(\mathbf{x}, ...)$ for the coordinates in Ω or Ω^δ.*

The equations that govern the propagation of electromagnetic waves in the cable Ω^δ are Maxwell's equations, whose unknowns are the two time dependent vector fields E^δ (the electric field) and H^δ (the magnetic field) ($\nabla\times$ is the 3D curl operator)

$$\left\{ \begin{array}{ll} \varepsilon^\delta \, \partial_t E^\delta + \sigma_e^\delta \, E^\delta - \nabla \times H^\delta = \mathbf{j}^\delta, & \mathbf{x} \in \Omega^\delta, \quad t > 0, \\ \mu^\delta \, \partial_t H^\delta + \sigma_m^\delta \, H^\delta + \nabla \times E^\delta = \mathbf{0}, & \mathbf{x} \in \Omega^\delta, \quad t > 0, \end{array} \right. \tag{46}$$

completed with perfectly conducting boundary conditions

$$E^\delta \times \mathbf{n}^\delta = \mathbf{0} \quad \mathbf{x} \in \partial\Omega^\delta, \quad t > 0, \tag{47}$$

and zero initial conditions:

$$E^\delta(\mathbf{x}, 0) = \mathbf{0}, \quad H^\delta(\mathbf{x}, 0) = \mathbf{0}, \quad \mathbf{x} \in \Omega^\delta. \tag{48}$$

The source current $\mathbf{j}^\delta$ is also defined by rescaling, $\mathbf{j}^\delta = \mathbf{j} \circ \mathcal{G}_\delta^{-1}$ where, for simplicity:

$$\mathbf{j} = (\, j_T, 0\,)^t, \quad j_T = (\, j_1, j_2\,)^t, \quad \operatorname{div} j_T = \partial_1 j_1 + \partial_2 j_2 = 0, \quad \mathbf{x} \in \Omega. \tag{49}$$

For the forthcoming asymptotic analysis, it is useful to work in the reference geometry Ω with the rescaled electromagnetic field:

$$\widetilde{E}^\delta = E^\delta \circ \mathcal{G}_\delta, \quad \widetilde{E}^\delta = E^\delta \circ \mathcal{G}_\delta \tag{50}$$

The longitudinal variable x_3, being invariant by $\mathcal{G}_\delta$, plays a different role compared to the transverse variables $x := (x_1, x_2)$, it is therefore natural to distinguish the transverse components of the electromagnetic fields from their longitudinal ones:

$$\widetilde{E}_T^\delta = (\widetilde{E}_1^\delta, \widetilde{E}_2^\delta)^t, \;\; \widetilde{H}_T^\delta = (\widetilde{H}_1^\delta, \widetilde{H}_2^\delta)^t \quad \text{from} \quad \widetilde{E}_3^\delta, \widetilde{H}_3^\delta.$$

We can rewrite (46) with these new unknowns and the notation of section 2 (note that a δ^{-1} factor appears where a transverse derivative, in x_1 or x_2, is involved):

$$\left\{\begin{array}{ll}\big(\varepsilon\,\partial_t+\sigma_e\big)\,\widetilde{E}^\delta_T-\partial_3\,\big(\mathbf{e}_3\times\widetilde{H}^\delta_T\big)-\delta^{-1}\,\mathbf{rot}\,\widetilde{H}^\delta_3=j_T, & \mathbf{x}\in\Omega,\quad t>0,\\[2mm] \big(\varepsilon\,\partial_t+\sigma_e\big)\,\widetilde{E}^\delta_3-\delta^{-1}\,\mathrm{rot}\,\widetilde{H}^\delta_T=0, & \mathbf{x}\in\Omega,\quad t>0,\\[2mm] \big(\mu\,\partial_t+\sigma_m\big)\,\widetilde{H}^\delta_T+\partial_3\,\big(\mathbf{e}_3\times\widehat{E}^\delta_T\big)+\delta^{-1}\,\mathbf{rot}\,\widetilde{E}_3=\mathbf{0}, & \mathbf{x}\in\Omega,\quad t>0,\\[2mm] \big(\mu\,\partial_t+\sigma_m\big)\,\widetilde{H}^\delta_3+\delta^{-1}\,\mathrm{rot}\,\widetilde{E}^\delta_T=0, & \mathbf{x}\in\Omega,\quad t>0.\end{array}\right. \tag{51}$$

as well as the boundary conditions (47) (multiplied by δ^{-1} for convenience)

$$\delta^{-1}\,\widetilde{E}^\delta_T\times n=\mathbf{0},\quad \delta^{-1}\,\widetilde{E}^\delta_3\,n-g\,\widetilde{E}^\delta_T=0. \tag{52}$$

In the next section, our goal is to analyze the behavior of the solution of (51, 52) when δ tends to 0.

4. **Asymptotic analysis.**

As in section 2, our (formal) developments in this section are fully justified provided that the functions we manipulate are smooth enough . This requires implicitly smoothness assumptions of the geometry of the cable and of the coefficients of the problem. However, we are convinced that these assumptions could be removed.

4.1. **A formal asymptotic expansion.**

The idea is to conjecture a formal power series expansion in δ of the solution of (51, 52). It is useful to pass to the frequency domain via a time Fourier transform:

$$\widetilde{E}^\delta(\mathbf{x},t)\;\longrightarrow\;\widehat{E}^\delta(\mathbf{x},\omega),\quad \widetilde{H}^\delta(\mathbf{x},t)\;\longrightarrow\;\widehat{H}^\delta(\mathbf{x},\omega). \tag{53}$$

Taking into account zero initial data, we obtain for each value of the frequency ω, the following problem for the Fourier transform of the electromagnetic field (we use the notation of section 2.1, indicate by the index T (resp. 3) the transverse (and resp. longitudinal) of a 3D vector field):

$$\left\{\begin{array}{lll}\big(i\omega\,\varepsilon+\sigma_e\big)\,\widehat{E}^\delta_T-\partial_3\,\big(\mathbf{e}_3\times\widehat{H}^\delta_T\big)-\delta^{-1}\,\mathbf{rot}\,\widehat{H}^\delta_3=j_T, & \mathbf{x}\in\Omega, & (a)\\[2mm] \big(i\omega\,\varepsilon+\sigma_e\big)\,\widehat{E}^\delta_3-\delta^{-1}\,\mathrm{rot}\,\widehat{H}^\delta_T=0, & \mathbf{x}\in\Omega, & (b)\\[2mm] \big(i\omega\,\mu+\sigma_m\big)\,\widehat{H}^\delta_T+\partial_3\,\big(\mathbf{e}_3\times\widehat{E}^\delta_T\big)+\delta^{-1}\,\mathbf{rot}\,\widehat{E}^\delta_3=\mathbf{0}, & \mathbf{x}\in\Omega, & (c)\\[2mm] \big(i\omega\,\mu+\sigma_m\big)\,\widehat{H}^\delta_3+\delta^{-1}\,\mathrm{rot}\,\widehat{E}^\delta_T=0, & \mathbf{x}\in\Omega, & (d)\end{array}\right. \tag{54}$$

with the boundary conditions (deduced from (52))

$$\delta^{-1}\,\widehat{E}^\delta_T\times n=\mathbf{0},\quad (a)\quad \delta^{-1}\,\widehat{E}^\delta_3\,n-g\,\widehat{E}^\delta_T=0.\quad (b) \tag{55}$$

We shall also use the "hidden" divergence free equations, that are easily deduced from (54) (see for instance [11])

$$\left\{\begin{array}{lll}\delta^{-1}\;\operatorname{div}\big((i\omega\,\varepsilon+\sigma_e)\widehat{E}^\delta_T\big)\;+\;\partial_{x_3}\big((i\omega\,\varepsilon+\sigma_e)\widehat{E}^\delta_3\big)=0, & \mathbf{x}\in\Omega, & (a)\\[2mm] \delta^{-1}\;\operatorname{div}\big((i\omega\,\mu+\sigma_m)\widehat{H}^\delta_T\big)\;+\;\partial_{x_3}\big((i\omega\,\mu+\sigma_m)\widehat{H}^\delta_3\big)=0, & \mathbf{x}\in\Omega, & (b)\end{array}\right. \tag{56}$$

and the "hidden" boundary condition for the magnetic field (see again [11])

$$\delta^{-1}\,\widehat{H}^\delta_T\cdot n-(\mathcal{V}\cdot n)\,\widehat{H}^\delta_3=0,\quad \mathbf{x}\in\partial\Omega. \tag{57}$$

Our approach consists in postulating a priori an asymptotic expansion in power series expansion of the form

$$\widehat{E}^\delta = \widehat{E}^0 + \delta\, \widehat{E}^1 + \delta^2\, \widehat{E}^2 + \cdots, \quad \widehat{H}^\delta = \widehat{H}^0 + \delta\, \widehat{H}^1 + \delta^2\, \widehat{H}^2 + \cdots, \tag{58}$$

and to identify and characterize the formal limit field $(\widehat{E}^0, \widehat{H}^0)$. For this, we substitute the expansions (58) into equations (54) to (57) and identify each power of δ in the resulting series, which begins with the δ^{-1} terms (because of the δ^{-1} factors in (54) to (57)).

Limit longitudinal fields

Concerning, the fields $\widehat{E}_3^0$ and $\widehat{H}_3^0$, it suffices to proceed as in [9] (we omit here the details) to show that

$$\widehat{E}_3^0 = \widehat{H}_3^0 = 0 \tag{59}$$

which means that the limit field is transversely polarized. This is considered as an assumption in many textbooks on transmission lines [12].

Identification of the δ^{-1} terms.

Equations (54-(d)), (56-(a)) and (55-(b)) lead to

$$\operatorname{rot} \widehat{E}_T^0 = 0, \quad \operatorname{div}\big((\varepsilon + \sigma_e/i\omega)\widehat{E}_T^0\big) = 0 \text{ in } \Omega, \quad \widehat{E}_T^0 \times n = 0, \text{ on } \partial\Omega. \tag{60}$$

Writing (60) for each x_3, considered as a parameter, one realizes that, according to section 2.2, we have

$$\forall\, x_3 \in \mathbb{R}, \quad \widehat{E}_T^0(.,x_3) \in \mathcal{E}\big(\widehat{\varepsilon}(.,x_3,\omega), S_{x_3}\big), \quad \widehat{\varepsilon}(.,x_3,\omega) := \varepsilon(.,x_3) + \frac{\sigma_e(.,x_3)}{i\omega}. \tag{61}$$

Therefore, setting

$$\widehat{\varphi}_{e,j}(\cdot,x_3,\omega) := \varphi_j\big(\widehat{\varepsilon}(.,x_3,\omega), S_{x_3}\big) \in H^1(S_{x_3}), \tag{62}$$

which defines N scalar fields $\widehat{\varphi}_{e,j}(x_3,\omega) : \Omega \mapsto \mathbb{C}$, we know that there exist N 1D (in space) complex valued functions $V_j(x_3,\omega)$ (electric potentials), such that:

$$\forall\, x_3 \in \mathbb{R}, \quad \widehat{E}_T^0(.,x_3,\omega) = \sum_{j=1}^N \widehat{V}_j(x_3,\omega)\, \nabla\widehat{\varphi}_{e,j}(\cdot,x_3,\omega). \tag{63}$$

In the same way, equations (54-(b)), (56-(b)) and (57) lead to

$$\operatorname{rot} \widehat{H}_T^0 = 0, \quad \operatorname{div}\big((\mu + \sigma_m/i\omega)\widehat{H}_T^0\big) = 0 \text{ in } \Omega, \quad \widehat{H}_T^0 \cdot n = 0 \text{ on } \partial\Omega. \tag{64}$$

Writing (60) for each x_3, one sees that, according to section ,

$$\forall\, x_3 \in \mathbb{R}, \ \widehat{H}_T^0(.,x_3) \in \mathcal{H}\big(\widehat{\mu}(.,x_3,\omega), S_{x_3}\big), \ \widehat{\mu}(.,x_3,\omega) := \mu(.,x_3) + \frac{\sigma_m(.,x_3)}{i\omega}. \tag{65}$$

Therefore, setting

$$\widehat{\psi}_{m,j}(\cdot,x_3,\omega) := \psi_j\big(\widehat{\mu}(.,x_3,\omega), S_{x_3}\big) \in H^1(S_{x_3}^\Gamma), \tag{66}$$

which defines N scalar fields $\widehat{\psi}_{m,j}(x_3,\omega) : \Omega \mapsto \mathbb{C}$, we know that there exist N 1D (in space) complex valued functions $I_j(x_3,\omega)$ (electric currents), such that:

$$\forall\, x_3 \in \mathbb{R}, \quad \widehat{H}_T^0(.,x_3,\omega) = \sum_{j=1}^N \widehat{I}_j(x_3,\omega)\, \widetilde{\nabla}\widehat{\psi}_{m,j}(\cdot,x_3,\omega). \tag{67}$$

Formulas (63) and (67) express a generalized separation of variables, valid at the limit $\delta \to 0$. The fields $\nabla\varphi_{e,j}(\cdot, x_3, \omega)$ and $\widetilde{\nabla}\psi_{m,j}(\cdot, x_3, \omega)$ can be precomputed by solving 2D boundary value problems in the cross sections S_{x_3}. It remains to determine the equations that determine the functions $V_j(x_3, \omega)$ and $I_j(x_3, \omega)$, which will provide us our effective model, the generalized telegrapher's model. For this we need to proceed to the next step of the identification process.

Identification of the δ^0 terms.

Using (63) and (67), the δ^0 terms in equations (54-(a)) and (54-(c)) are

$$\left\{\begin{array}{ll}(i\omega\,\varepsilon + \sigma_e)\displaystyle\sum_{j=1}^{N} \widehat{V}_j\,\nabla\widehat{\varphi}_{e,j} - \partial_3\Big(\mathbf{e}_3 \times \sum_{j=1}^{N} \widehat{I}_j\,\nabla\widehat{\psi}_{m,j}\Big) - \mathbf{rot}\,\widehat{H}_3^1 = \widehat{j}_T, & (a)\\ (i\omega\,\mu + \sigma_m)\displaystyle\sum_{j=1}^{N} \widehat{I}_j\,\nabla\widehat{\psi}_{m,j} + \partial_3\Big(\mathbf{e}_3 \times \sum_{j=1}^{N} \widehat{V}_j\,\nabla\widehat{\varphi}_{e,j}\Big) + \mathbf{rot}\,\widehat{E}_3^1 = 0, & (b)\end{array}\right. \tag{68}$$

while (55) gives

$$\widehat{E}_3^1\,n - g\,\widehat{E}_T^0 = 0 \tag{69}$$

To get rid of the (unknown) terms $\widehat{H}_3^1$ and $\widehat{E}_3^1$, we take the L^2 scalar product, in each cross section S_{x_3}, of (68-(a)) with each $\nabla\widehat{\varphi}_{e,i}$ and of (68-(b)) with each $\widetilde{\nabla}\widehat{\psi}_{m,i}$.

This is motivated by the following identities (70) and (71).

First, using Green's formula, we have, in each section S_{x_3}

$$\Big(\mathbf{rot}\,\widehat{H}_3^1, \nabla\widehat{\varphi}_{e,i}\Big)_{L^2(S_{x_3})} = \int_{\partial S_{x_3}} \widehat{H}_3^1\;\overline{\nabla\widehat{\varphi}_{e,i} \times n} = 0, \tag{70}$$

since $\nabla\widehat{\varphi}_{e,i} \times n = 0$ along ∂S_{x_3} and $\mathrm{rot}\,\nabla\widehat{\varphi}_{e,i} = 0$.

On the other hand, thanks to the jump relations satisfies by $\widehat{\psi}_{m,i}$, the reader will easily verify that

$$\mathbf{rot}\;\widetilde{\nabla}\widehat{\psi}_{m,i} = 0 \text{ in } S_{x_3}.$$

Therefore, by Green's formula

$$\Big(\mathbf{rot}\,\widehat{E}_3^1, \widetilde{\nabla}\widehat{\psi}_{m,i}\Big)_{L^2(S_{x_3})} = \int_{\partial S_{x_3}} \widehat{E}_3^1\;\overline{\widetilde{\nabla}\widehat{\psi}_{m,i} \times n}.$$

Then, using (69), we have, along ∂S_{x_3} :

$$\begin{aligned}\widehat{E}_3^1\,\overline{\widetilde{\nabla}\widehat{\psi}_{m,i} \times n} \equiv -\,\widehat{E}_3^1\;\overline{\widetilde{\mathbf{rot}}\,\widehat{\psi}_{m,i}\cdot n} \quad &= \quad -\,g\,\big(\widehat{E}_T^0\cdot n\big)\big(\overline{\widetilde{\mathbf{rot}}\;\widehat{\psi}_{m,i}\cdot n}\big)\\ &= \quad -\,g\,\widehat{E}_T^0\cdot\overline{\widetilde{\mathbf{rot}}\;\widehat{\psi}_{m,i}},\end{aligned}$$

where the last equality results from the fact that $\partial_n\widehat{\psi}_{m,i} = 0$ means that $\widetilde{\mathbf{rot}}\;\widehat{\psi}_{m,i}$ is normal to ∂S_{x_3}. Consequently, using Green's formula and (63)

$$\begin{aligned}\Big(\mathbf{rot}\,\widehat{E}_3^1, \widetilde{\nabla}\widehat{\psi}_{m,i}\Big)_{L^2(S_{x_3})} \quad &= \quad -\int_{S_{x_3}} g\,\widehat{E}_T^0\cdot\overline{\widetilde{\mathbf{rot}}\;\widehat{\psi}_{m,i}}\\ &= \quad -\sum_{j=1}^{N}\Big(\int_{\partial S_{x_3}} g\,\nabla\widehat{\varphi}_{e,j}\cdot\overline{\widetilde{\mathbf{rot}}\;\widehat{\psi}_{m,i}}\;\Big)\widehat{V}_j\;.\end{aligned} \tag{71}$$

For the second terms in (68-(a) and (b)), we shall use the following lemma, whose proof will be omitted since it is a straightforward adaptation of lemma 3.4 in [9]

Lemma 4.1.

$$(i) \quad \Big(\partial_3 \,\mathbf{rot}\, \widehat{\varphi}_{e,j}, \widetilde{\nabla} \widehat{\psi}_{m,i}\Big)_{L^2(S_{x_3})} + \Big(\int_{\partial S_{x_3}} g\, \nabla \widehat{\varphi}_{e,j} \cdot \overline{\widetilde{\mathbf{rot}}\, \widehat{\psi}_{m,i}}\Big) = 0.$$

$$(ii) \quad \Big(\partial_3 \,\widetilde{\mathbf{rot}}\, \widehat{\psi}_{m,j}, \nabla \varphi_{e,i}\Big)_{L^2(S^\Gamma_{x_3})} = 0.$$

Using $\mathbf{e}_3 \times \nabla = -\mathbf{rot}$, the bi-orthogonality relations (21) and lemma 4.1, we have

$$- \Big(\partial_3 \big(\mathbf{e}_3 \times \sum_{j=1}^N \widehat{I}_j\, \widetilde{\nabla} \widehat{\psi}_{m,j}\big), \nabla \widehat{\varphi}_{e,i}\Big)_{L^2(S_{x_3})} = \partial_3 I_i \tag{72}$$

$$\begin{aligned} \Big(\partial_3 \big(\mathbf{e}_3 \times \sum_{j=1}^N \widehat{V}_j\, \nabla \widehat{\varphi}_{e,j}\big), \widetilde{\nabla} \widehat{\psi}_{m,i}\Big)_{L^2(S_{x_3})} &= \partial_3 V_i \\ &+ \sum_{j=1}^N \Big(\int_{\partial S_{x_3}} g\, \nabla \widehat{\varphi}_{e,j} \cdot \overline{\widetilde{\mathbf{rot}}\, \widehat{\psi}_{m,i}}\Big) \widehat{V}_j. \end{aligned} \tag{73}$$

Finally, taking the L^2 scalar product in S_{x_3} of (68-(a)) with $\nabla \widehat{\varphi}_{e,i}$ and (68-(b)) with $\widetilde{\nabla} \widehat{\psi}_{m,i}$, we obtain, using (70, 71, 72, 73), the equations for the potentials $\widehat{V}_j$ and the currents $\widehat{I}_j$.

To write this model in a compact way, we introduce the vectors

$$\widehat{\mathbf{V}}(x_3,\omega) := \big(\widehat{V}_j(x_3,\omega)\big)_{1\le j\le N}, \quad \widehat{\mathbf{I}}(x_3,\omega) := \big(\widehat{I}_j(x_3,\omega)\big)_{1\le j\le N} \tag{74}$$

and define the matrices (with the notation (22) and (23) of section 2)

$$\widehat{\mathbf{Y}}_e(x_3,\omega) := \widehat{\mathbf{Y}}(\widehat{\varepsilon}(\cdot,x_3,\omega), S_{x_3}), \quad \widehat{\mathbf{Z}}_m(x_3,\omega) := \widehat{\mathbf{Z}}(\widehat{\mu}(\cdot,x_3,\omega), S_{x_3}), \tag{75}$$

where $\widehat{\varepsilon}$ and $\widehat{\mu}$ have been defined in (61) and (65). One then gets

$$\left\{ \begin{array}{ll} i\omega\, \widehat{\mathbf{Y}}_e(x_3,\omega)\, \widehat{\mathbf{V}}(x_3,\omega) + \partial_3\, \widehat{\mathbf{I}}(x_3,\omega) = \widehat{\mathbf{J}}_T(x_3,\omega), & (a) \\[2mm] i\omega\, \widehat{\mathbf{Z}}_m(x_3,\omega)\, \widehat{\mathbf{I}}(x_3,\omega) + \partial_3\, \widehat{\mathbf{V}}(x_3,\omega) = 0, & (b) \end{array} \right. \tag{76}$$

where the vector of sources is defined by

$$\widehat{\mathbf{J}}_T(x_3,\omega) = \big(\widehat{J}_{T,i}(x_3,\omega)\big)_{1\le i\le N}, \quad \widehat{J}_{T,i}(x_3,\omega) = \int_{S_{x_3}} \widehat{j}_T \cdot \overline{\nabla \widehat{\varphi}_{e,i}(\omega)}. \tag{77}$$

4.2. **Derivation of the generalized telegrapher's model in time domain.** Our objective is to write the time domain formulation of equation (76). This is done formally by an inverse Fourier transform that can be applied on (76). We shall do it in a way that emphasizes energy preservation (or dissipation) relations and related stability properties of the resulting evolution problem. Towards this goal, we introduce the high frequency behavior of the functions

$$\widehat{\varphi}_{e,j} \text{ and } \widehat{\psi}_{m,j}.$$

These high frequency limits are naturally defined after noticing that

$$\widehat{\varepsilon} \to \varepsilon \text{ and } \widehat{\mu} \to \mu \text{ when } \omega \to \pm\infty.$$

This justifies to introduce

$$\varphi_{e,j}^{\infty}(\cdot,x_3) := \varphi_j\big(\varepsilon(.,x_3),S_{x_3}\big), \quad \psi_{e,j}^{\infty}(\cdot,x_3) := \psi_j\big(\mu(.,x_3),S_{x_3}\big), \tag{78}$$

as well as the (frequency dependent) residual terms

$$\begin{aligned} \widehat{\varphi}_{e,j}^{r}(\cdot,x_3,\omega) &:= \widehat{\varphi}_{e,j}(\cdot,x_3,\omega) - \varphi_{e,j}^{\infty}(\cdot,x_3), \\ \widehat{\psi}_{m,j}^{r}(\cdot,x_3,\omega) &:= \widehat{\psi}_{m,j}(\cdot,x_3,\omega) - \psi_{m,j}^{\infty}(\cdot,x_3). \end{aligned} \tag{79}$$

By definition of $\varphi_{e,j}^{\infty}$, we remark that $\widehat{\varphi}_{e,j}^{r} \in H^1(S_{x_3})$ is the unique solution of

$$\left\{ \begin{array}{ll} \operatorname{div}\big(\widehat{\varepsilon}\,\nabla\widehat{\varphi}_{e,j}^{r}\big) = -\dfrac{1}{i\omega}\operatorname{div}\big(\,\sigma_e\,\nabla\varphi_{e,j}^{\infty}\,\big), & \text{in } S, \\ \widehat{\varphi}_{e,j}^{r} = 0, & \text{on } \partial S. \end{array} \right. \tag{80}$$

Multiplying this equation by $\overline{\varphi_{e,j}^{r}}$, we get the relation

$$\int_{S_{x_3}} (i\omega\varepsilon + \sigma_e)\,|\nabla\,\widehat{\varphi}_{e,j}^{r}|^2 = -\int_{S_{x_3}} \sigma_e\,\nabla\,\widehat{\varphi}_{e,j}^{\infty}\cdot\overline{\nabla\,\widehat{\varphi}_{e,j}^{r}},$$

which enables us to deduce the following estimates (by simply taking the imaginary and the real part of the previous equality)

$$\big\|\nabla\,\widehat{\varphi}_{e,j}^{r}\big\|_{L^2(S_{x_3})} \leq \frac{1}{\varepsilon_-|\omega|}\,\big\|\sigma_e\,\nabla\,\varphi_{e,j}^{\infty}\big\|_{L^2(S_{x_3})}, \tag{81}$$

$$\Big\|\sigma_e^{1/2}\,\nabla\,\widehat{\varphi}_{e,j}^{r}\Big\|_{L^2(S_{x_3})} \leq \Big\|\sigma_e^{1/2}\,\nabla\,\varphi_{e,j}^{\infty}\Big\|_{L^2(S_{x_3})}. \tag{82}$$

Similarly, the residual function $\widehat{\psi}_{m,j}^{r} \in H^1(S_{x_3})$ is the unique solution of

$$\left\{ \begin{array}{ll} \operatorname{div}\big(\widehat{\mu}\,\nabla\widehat{\psi}_{m,j}^{r}\big) = -\dfrac{1}{i\omega}\operatorname{div}\big(\,\sigma_m\,\widetilde{\nabla}\psi_{m,j}^{\infty}\,\big), & \text{in } S, \\ \nabla\widehat{\psi}_{m,j}^{r}\cdot n = 0, & \text{on } \partial S, \end{array} \right. \tag{83}$$

and satisfies

$$\Big\|\nabla\,\widehat{\psi}_{m,j}^{r}\Big\|_{L^2(S_{x_3})} \leq \frac{1}{\mu_-|\omega|}\,\Big\|\sigma_m\,\widetilde{\nabla}\,\psi_{m,j}^{\infty}\Big\|_{L^2(S_{x_3})},$$

$$\Big\|\sigma_m^{1/2}\,\nabla\,\widehat{\psi}_{m,j}^{r}\Big\|_{L^2(S_{x_3})} \leq \Big\|\sigma_m^{1/2}\,\widetilde{\nabla}\,\psi_{m,j}^{\infty}\Big\|_{L^2(S_{x_3})}.$$

The high frequency decomposition (79) yields the following proposition

Proposition 1. *The matrices $\widehat{\mathbf{Y}}_e(x_3,\omega)$ and $\widehat{\mathbf{Z}}_m(x_3,\omega)$ satisfy*

$$i\omega\,\widehat{\mathbf{Y}}_e(x_3,\omega) = i\omega\,\mathbf{C}_\infty(x_3) + \mathbf{G}_\infty(x_3) + \widehat{\mathbf{K}}_e(x_3,\omega), \tag{84}$$

$$i\omega\,\widehat{\mathbf{Z}}_m(x_3,\omega) = i\omega\,\mathbf{L}_\infty(x_3) + \mathbf{R}_\infty(x_3) + \widehat{\mathbf{K}}_m(x_3,\omega), \tag{85}$$

where, the $N \times N$ real symmetric non negative matrices $(\mathbf{C}_\infty, \mathbf{G}_\infty, \mathbf{L}_\infty, \mathbf{R}_\infty)$ are defined by

$$\begin{aligned}(\mathbf{C}_\infty)_{ij} &:= \int_{S_{x_3}} \varepsilon\, \nabla\varphi_{e,i}^\infty \cdot \nabla\varphi_{e,j}^\infty\, dx, \quad &(\mathbf{G}_\infty)_{ij} &:= \int_{S_{x_3}} \sigma_e\, \nabla\varphi_{e,i}^\infty \cdot \nabla\varphi_{e,j}^\infty\, dx,\\ (\mathbf{L}_\infty)_{ij} &:= \int_{S_{x_3}} \mu\, \widetilde{\nabla}\psi_{m,i}^\infty \cdot \widetilde{\nabla}\psi_{m,j}^\infty\, dx, \quad &(\mathbf{R}_\infty)_{ij} &:= \int_{S_{x_3}} \sigma_m\, \widetilde{\nabla}\psi_{m,i}^\infty \cdot \widetilde{\nabla}\psi_{m,j}^\infty\, dx.\end{aligned} \tag{86}$$

The matrices $\mathbf{C}^\infty$ and $\mathbf{L}^\infty$ are positive definite, $\mathbf{C}^\infty$ is a strict M-matrix and $\mathbf{L}^\infty$ has strictly positive entries. Finally, the frequency dependent symmetric matrices $(\widehat{\mathbf{K}}_e, \widehat{\mathbf{K}}_m)$ are defined by

$$(\widehat{\mathbf{K}}_e)_{ij} = \int_{S_{x_3}} \sigma_e\, \nabla\widehat{\varphi}_{e,i}^r \cdot \nabla\varphi_{e,j}^\infty\, dx, \quad (\widehat{\mathbf{K}}_m)_{ij} = \int_{S_{x_3}} \sigma_m\, \nabla\widehat{\psi}_{m,i}^r \cdot \widetilde{\nabla}\psi_{m,j}^\infty\, dx. \tag{87}$$

Proof. We prove only (84), the argument being similar to prove (85). From the definition (22, 75) of $\widehat{\mathbf{Y}}_e(x_3,\omega)$ and the intermediate result (24) we have

$$\widehat{\mathbf{Y}}_e = \int_{S_{x_3}} \widehat{\varepsilon}\, \nabla\widehat{\varphi}_{e,i} \cdot \nabla\widehat{\varphi}_{e,j}\, dx = \int_{S_{x_3}} \widehat{\varepsilon}\, \nabla\widehat{\varphi}_{e,i} \cdot \nabla\varphi_{e,j}^\infty\, dx,$$

where the last equality is obtained using Remark 3 with $\rho = \widehat{\varepsilon}$ and $\varphi^* = \widehat{\varphi}_{e,j}^r$. Using again Remark 3 with $\rho = \varepsilon$ and $\varphi^* = \widehat{\varphi}_{e,i}^r$, we can simply further the previous expression:

$$\widehat{\mathbf{Y}}_e = \int_{S_{x_3}} \varepsilon\, \nabla\varphi_{e,i}^\infty \cdot \nabla\varphi_{e,j}^\infty\, dx + \int_{S_{x_3}} \frac{\sigma_e}{i\omega}\, \nabla\widehat{\varphi}_{e,i}^r \cdot \nabla\varphi_{e,j}^\infty\, dx,$$

which conclude the proof of the decomposition. The symmetry property of $(\widehat{\mathbf{K}}_e, \widehat{\mathbf{K}}_m)$ is a direct consequence of the obvious symmetry of $(\mathbf{C}_\infty, \mathbf{G}_\infty, \mathbf{L}_\infty, \mathbf{R}_\infty)$ and of $(\widehat{\mathbf{Y}}_e, \widehat{\mathbf{Z}}_m)$ (as proven in lemma 2.2). The properties of the matrices $\mathbf{C}^\infty$ and $\mathbf{L}^\infty$ are straightforward application of lemma 2.6. $\square$

Lemma 4.2. *The function of the frequency $\big((\widehat{\mathbf{K}}_e)_{ij}(x_3,\cdot), (\widehat{\mathbf{K}}_m)_{ij}(x_3,\cdot)\big)$ satisfy*

$$\widehat{\mathbf{K}}_e(x_3,\cdot) \in L^2\big(\mathbb{R}, \mathcal{M}_N(\mathbb{C})\big), \quad \widehat{\mathbf{K}}_m(x_3,\cdot) \in L^2\big(\mathbb{R}, \mathcal{M}_N(\mathbb{C})\big),$$

and, for all $u \in \mathbb{C}^N$,

$$\begin{aligned}\inf_{\omega\in\mathbb{R}} \big(\mathcal{R}e\big[\mathbf{G}_\infty(x_3) + \widehat{\mathbf{K}}_e(x_3,\cdot)\big]u, u\big)_{\mathbb{C}^N} \geq 0,\\ \inf_{\omega\in\mathbb{R}} \big(\mathcal{R}e\big[\mathbf{R}_\infty(x_3) + \widehat{\mathbf{K}}_m(x_3,\cdot)\big]u, u\big)_{\mathbb{C}^N} \geq 0.\end{aligned}$$

Proof. Using (87), Cauchy-Schwartz inequality and estimate (81), we get

$$|(\widehat{\mathbf{K}}_e)_{ij}| \leq \big\|\nabla\, \widehat{\varphi}_{e,j}^r\big\|_{L^2(S_{x_3})} \big\|\sigma_e\, \nabla\, \varphi_{e,j}^\infty\big\|_{L^2(S_{x_3})} \leq \frac{1}{\mu_-|\omega|} \big\|\sigma_e\, \nabla\, \varphi_{e,j}^\infty\big\|_{L^2(S_{x_3})}^2.$$

In a similar way, using (82) we can deduce an uniform, frequency independent, bound on $(\widehat{\mathbf{K}}_e)_{ij}(x_3,\cdot)$:

$$|(\widehat{\mathbf{K}}_e)_{ij}| \leq \Big\|\sigma_e^{1/2}\, \nabla\, \widehat{\varphi}_{e,j}^r\Big\|_{L^2(S_{x_3})} \Big\|\sigma_e^{1/2}\, \nabla\, \varphi_{e,j}^\infty\Big\|_{L^2(S_{x_3})} \leq \Big\|\sigma_e^{1/2}\, \nabla\, \varphi_{e,j}^\infty\Big\|_{L^2(S_{x_3})}^2.$$

Combining the two previous estimates we deduce the first result of the lemma which can be proven for $\mathbf{K}_m$ in a very similar way. Next the positivity properties follow from the simple observation that

$$\mathcal{R}e[\mathbf{G}_\infty(x_3) + \widehat{\mathbf{K}}_e(x_3,\cdot)]_{ij} = \mathcal{R}e[i\omega\,\widehat{\mathbf{Y}}_e]_{ij} = \int_{S_{x_3}} \sigma_e\,\nabla\widehat{\varphi}_{e,i}\cdot\overline{\nabla\widehat{\varphi}_{e,j}}\,dx,$$

which define a positive matrix since $\sigma_e \geq 0$ (the argument uses the expression (25) of the scalar product). □

Looking at (80) suggests to introduce the causal function $\varphi^r_{e,j}(x,x_3,t)$ defined by $\varphi^r_{e,j}(x_3,x,t) = 0$ for $t<0$ and is solution, for $t \geq 0$, of the Cauchy problem

$$\left\{\begin{array}{ll} \operatorname{div}\Big(\varepsilon\dfrac{\partial}{\partial t}\nabla\varphi^r_{e,j} + \sigma_e\nabla\varphi^r_{e,j}\Big) = 0, & \text{in } S_{x_3}, \\ \varphi^r_{e,j} = 0, & \text{on } \partial S_{x_3}, \\ \operatorname{div}\big(\varepsilon\,\nabla\varphi^r_{e,j}\big) = -\operatorname{div}\big(\sigma_e\,\nabla\varphi^\infty_{e,j}\big), & \text{in } S_{x_3} \text{ at } t=0. \end{array}\right. \tag{88}$$

The well-posedness of (88) is proved as in [9], Section 5, with

$$\varphi^r_{e,j}(\cdot,x_3,\cdot) \in C^\infty(\mathbb{R}^+; H^1_0(S_{x_3})).$$

These results are due to the fact that (88) can be rewritten as an abstract evolution problem in $H^1_0(S_{x_3})$ of the form (see [9])

$$\frac{d}{dt}\,\varphi^r_{e,j}(\cdot,x_3,\cdot) + \mathbf{A}_e(x_3)\,\varphi^r_{e,j}(\cdot,x_3,\cdot) = 0, \quad \mathbf{A}_e(x_3) \in \mathcal{L}\big(H^1_0(S_{x_3})\big).$$

With the same arguments the function $\psi^r_{m,j}(x_3,x,t)$ is the causal function uniquely defined for $t>0$ as the solution of the evolution problem:

$$\left\{\begin{array}{ll} \operatorname{div}\Big(\mu\dfrac{\partial}{\partial t}\nabla\psi^r_{m,j} + \sigma_m\nabla\psi^r_{m,j}\Big) = 0, & \text{in } S_{x_3}, \\ \nabla\psi^r_{m,j}\cdot n = 0, & \text{on } \partial S_{x_3}, \\ \operatorname{div}\big(\mu\,\nabla\psi^r_{m,j}\big) = -\operatorname{div}\big(\sigma_m\,\widetilde{\nabla}\psi^\infty_{m,j}\big), & \text{in } S_{x_3} \text{ at } t=0. \end{array}\right. \tag{89}$$

which satisfies

$$\psi^r_{m,j}(\cdot,x_3,\cdot) \in C^\infty(\mathbb{R}^+; H^1(S_{x_3})).$$

We can now deduce a convenient time domain formulation of the generalized telegrapher's model from the equations (76) with the coefficients decomposed following proposition 1. Using the standard properties of the Fourier transform with respect to convolution, we obtain

$$\left\{\begin{array}{l} \mathbf{C}_\infty(x_3)\,\partial_t\mathbf{V} + \mathbf{G}_\infty(x_3)\,\mathbf{V} + \displaystyle\int_0^t \mathbf{K}_e(x_3,t-s)\,\mathbf{V}(x_3,s)\,ds + \partial_3\,\mathbf{I} = \mathbf{J}_T, \\ \mathbf{L}_\infty(x_3)\,\partial_t\mathbf{I} + \mathbf{R}_\infty(x_3)\,\mathbf{I} + \displaystyle\int_0^t \mathbf{K}_m(x_3,t-s)\,\mathbf{I}(x_3,s)\,ds + \partial_3\,\mathbf{V} = 0, \end{array}\right. \tag{90}$$

this set of equations being completed with zero initial conditions

$$\mathbf{V}(x_3,0) = \mathbf{I}(x_3,0) = 0. \tag{91}$$

In (90) the convolution kernels

$$(\mathbf{K}_e(x_3,\cdot), \mathbf{K}_m(x_3,\cdot)) \in L^2\big(\mathbb{R}^+; \mathcal{M}_N(\mathbb{R})\big)^2$$

are the inverse Fourier transform in time of $(\widehat{\mathbf{K}}_e(x_3,\cdot),\widehat{\mathbf{K}}_m(x_3,\cdot))$. According to (87), it is easy to see that they are given by the following explicit expressions

$$\begin{aligned}(\mathbf{K}_e)_{ij}(x_3,t) &= \int_{S_{x_3}} \sigma_e(x,x_3)\,\nabla\varphi^r_{e,i}(x,x_3,t)\cdot\nabla\varphi^\infty_{e,j}(x,x_3)\,dx,\\ (\mathbf{K}_m)_{ij}(x_3,t) &= \int_{S_{x_3}} \sigma_m(x,x_3)\,\nabla\psi^r_{m,i}(x,x_3,t)\cdot\widetilde{\nabla}\psi^\infty_{m,j}(x,x_3)\,dx.\end{aligned} \tag{92}$$

The model (90) is our effective 1D model in which:

- the effective coefficients $\mathbf{C}_\infty(x_3),\mathbf{G}_\infty(x_3),\mathbf{L}_\infty(x_3),\mathbf{R}_\infty(x_3)$ are deduced, through (86), from the solution of 2D elliptic problems in S_{x_3}, see (18, 19, 78),
- the kernels $(\mathbf{K}_e(x_3,\cdot),\mathbf{K}_m(x_3,\cdot))$ are deduced, through formulas (92), from the solution of 2D evolution problems in S_{x_3}, namely (88) and (89).

Finally it is worthwhile mentioning that from the solution $(\mathbf{V},\mathbf{I})$, one can reconstruct (approximately) the 3D electromagnetic field through the following formulas, deduced from formulas (63, 67) and the decompositions (79):

$$\begin{aligned}E^\delta(x,x_3,t) &\sim \sum_{j=1}^N \nabla\varphi^\infty_{e,j}\Big(\frac{x}{\delta},x_3,t\Big)\,V_j(x_3,t)\\ &\quad+\sum_{j=1}^N\int_0^t \nabla\varphi^r_{e,j}\Big(\frac{x}{\delta},x_3,t-s\Big)\,V_j(x_3,s)\,ds,\\ H^\delta(x,x_3,t) &\sim \sum_{j=1}^N \widetilde{\nabla}\psi^\infty_{m,j}\Big(\frac{x}{\delta},x_3,t\Big)\,I_j(x_3,t)\\ &\quad+\sum_{j=1}^N\int_0^t \nabla\psi^r_{m,j}\Big(\frac{x}{\delta},x_3,t-s\Big)\,I_j(x_3,s)\,ds.\end{aligned} \tag{93}$$

Remark 5. *We call a generalized telegrapher's model because the unknowns* $\mathbf{I}$ *and* $\mathbf{V}$ *are vector valued, and not scalar, and because of the time convolution terms that represent memory effects. Let us emphasize that the presence of these memory terms is due to the conjugated presence of losses in the medium and heterogeneity of the cross section. More precisely, as in [9], it is easy to show that:*

$$\begin{aligned}\mathbf{K}_e(x_3,\cdot)=0 &\iff \big(\sigma_e/\varepsilon\big)(x_3,\cdot) \text{ is constant in } S_{x_3},\\ \mathbf{K}_m(x_3,\cdot)=0 &\iff \big(\sigma_m/\mu\big)(x_3,\cdot) \text{ is constant in } S_{x_3}.\end{aligned} \tag{94}$$

5. Properties of the generalized telegrapher's model.

5.1. Well-posedness of the limit model.

Theorem 5.1. *If the source term as the regularity*

$$\mathbf{J}_T\in L^1(\mathbb{R}^+,L^2(\mathbb{R}^3))$$

and there exist strictly positive constants C_- *and* L_- *such that, for all* $u\in\mathbb{R}$

$$\inf_{x_3\in\mathbb{R}}(\mathbf{C}_\infty(x_3)u,u)_{\mathbb{R}^N}\ge C_-\,|u|^2_{\mathbb{R}^N},\quad \inf_{x_3\in\mathbb{R}}(\mathbf{L}_\infty(x_3)u,u)_{\mathbb{R}^N}\ge L_-\,|u|^2_{\mathbb{R}^N}, \tag{95}$$

then, there exists a unique weak solution $(\mathbf{V},\mathbf{I})\in C^0\big(\mathbb{R}^+,L^2(\mathbb{R})\big)^2$ *of (90, 91) and*

$$\|\mathbf{V}(\cdot,t)\|^2_{L^2(\mathbb{R})}+\|\mathbf{I}(\cdot,t)\|^2_{L^2(\mathbb{R})}\le\frac{1}{\min(C_-,L_-)}\int_0^t\|\mathbf{J}_T(\cdot,s)\|_{L^2(\mathbb{R})}ds. \tag{96}$$

The proof of the above result relies on the introduction of the following two bilinear forms in $L^2\big(0,t;L^2(\mathbb{R}^3)\big)$:

$$\begin{aligned}\mathbf{a}_e\big(t;\mathbf{U},\widetilde{\mathbf{U}}\big) &= \int_0^t\int_{\mathbb{R}}\big(\mathbf{G}_\infty(x_3)\,\mathbf{U}(x_3,s),\widetilde{\mathbf{U}}(x_3,s)\big)_{\mathbb{R}^N}\,dx_3\,ds\\ &\quad+\int_0^t\int_0^s\int_{\mathbb{R}}\big(\mathbf{K}_e(x_3,s-r)\,\mathbf{U}(x_3,r),\widetilde{\mathbf{U}}(x_3,s)\big)_{\mathbb{R}^N}\,dx_3\,dr\,ds,\end{aligned}$$

$$\begin{aligned}\mathbf{a}_m\big(t;\mathbf{U},\widetilde{\mathbf{U}}\big) &= \int_0^t\int_{\mathbb{R}}\big(\mathbf{R}_\infty(x_3)\,\mathbf{U}(x_3,s),\widetilde{\mathbf{U}}(x_3,s)\big)_{\mathbb{R}^N}\,dx_3\,ds\\ &\quad+\int_0^t\int_0^s\int_{\mathbb{R}}\big(\mathbf{K}_m(x_3,s-r)\,\mathbf{U}(x_3,r),\widetilde{\mathbf{U}}(x_3,s)\big)_{\mathbb{R}^N}\,dx_3\,dr\,ds,\end{aligned}$$

that have the important positivity property:

Lemma 5.2. *The bilinear forms $\mathbf{a}_e\big(t;\cdot,\cdot\big)$ and $\mathbf{a}_m\big(t;\cdot,\cdot\big)$ satisfy:*

$$\forall\ \mathbf{U}\in L^2\big(0,t;L^2(\mathbb{R}^3)\big),\qquad \mathbf{a}_\ell\big(t;\mathbf{U},\mathbf{U}\big)\geq 0,\qquad \ell=e,m. \tag{97}$$

Proof. We give the proof for $\mathbf{a}_e$. We extend $\mathbf{U}$ outside $[0,t]$ into a function $\mathbf{U}^\star$ such that

$$\mathbf{U}^\star(\cdot,s)=\mathbf{U}(\cdot,s)\text{ for }s\in[0,t]\text{ and }\mathbf{U}^\star(\cdot,s)=0\text{ otherwise},$$

then we have

$$\begin{aligned}\mathbf{a}_e\big(t;\mathbf{U},\mathbf{U}\big) &= \int_{\mathbb{R}}\int_{\mathbb{R}}(\mathbf{G}_\infty(x_3)\,\mathbf{U}^\star(x_3,s),\mathbf{U}^\star(x_3,s))_{\mathbb{R}^N}\,dx_3\,ds\\ &\quad+\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}(\mathbf{K}_e(x_3,s-r)\,\mathbf{U}^\star(x_3,r),\mathbf{U}^\star(x_3,s))_{\mathbb{R}^N}\,dx_3\,dr\,ds.\end{aligned}$$

Denoting $\widehat{\mathbf{U}}^\star$ the time Fourier transform of $\mathbf{U}^\star$, we get, by Parseval identity

$$\mathbf{a}_e\big(t;\mathbf{U},\mathbf{U}\big)=\int_{\mathbb{R}}\int_{\mathbb{R}}\big(\mathcal{R}e\,[\mathbf{G}_\infty(x_3)+\widehat{\mathbf{K}}_e(x_3,\omega)]\,\widehat{\mathbf{U}}^\star(x_3,\omega),\widehat{\mathbf{U}}^\star(x_3,\omega)\big)_{\mathbb{C}^N}\,d\omega\,dx_3.$$

We then conclude thanks to lemma 4.2. □

Proof of theorem 5.1. We first assume that j_T is regular enough and that a corresponding regular and integrable solution of (90, 91) exists. We multiply the two equations of (90) by $\mathbf{V}$ and $\mathbf{I}$ respectively, integrate in space over $\mathbb{R}$ and in time between 0 and t and add the two resulting equalities. We use (95) then to obtain the following inequality:

$$\begin{aligned}\frac{C_-}{2}\,\|\mathbf{V}(\cdot,t)\|^2_{L^2(\mathbb{R})}+\frac{L_-}{2}\,\|\mathbf{I}(\cdot,t)\|^2_{L^2(\mathbb{R})}\,+\,\mathbf{a}_e\big(t;\mathbf{V},\mathbf{V}\big)\,+\,\mathbf{a}_m\big(t;\mathbf{I},\mathbf{I}\big)&\\ \leq\int_0^t\int_{\mathbb{R}}(\mathbf{J}_T,\mathbf{V})_{\mathbb{R}^N}\,dx_3\,ds,&\end{aligned} \tag{98}$$

Lemma 5.2 then gives

$$\frac{C_-}{2}\,\|\mathbf{V}\|^2_{L^2(\mathbb{R})}+\frac{L_-}{2}\,\|\mathbf{I}\|^2_{L^2(\mathbb{R})}\leq\int_0^t\|\mathbf{J}_T(\cdot,t)\|_{L^2(\mathbb{R})}\,\|\mathbf{V}\|_{L^2(\mathbb{R})}\,ds,$$

and Gronwall's lemma enables us to derive the energy estimate (96). This estimate is the key tool to deduce existence and uniqueness of the solution in the right functional space. We shall not reproduce the (rather standard) proof but refer the reader to [5]. □

Theorem 5.1 guarantees the well-posedness of the generalized telegrapher's equation under the assumptions (95). In the next section, we investigate the assumptions to be done on the original 3D problem (these are not straightforward) in order that the effective coefficients given by (86) do satisfy this assumption.

5.2. Bounds for the effective coefficients.

Lemma 5.3. *(i) Assume that there exists a simple connected bounded domain $\mathcal{B}^*$ such that, for each $x_3 \in \mathbb{R}$, there exists a conformal mapping $\mathcal{T}_{c,x_3}$ such that*

$$\mathcal{T}_{c,x_3}(S_{x_3}) \subset \mathcal{B}^* \quad \text{and} \quad m_* := \inf_{x_3 \in \mathbb{R}} \inf_{1 \le j \le N} \text{meas } \mathcal{O}_j^{x_3} > 0 \tag{99}$$

then

$$\begin{aligned} (\mathbf{C}_\infty(x_3)u, u)_{\mathbb{C}^N} &\ge \varepsilon^- \, \lambda_D(\mathcal{B}^*) \, m_* \, |u|^2_{\mathbb{C}^N}, \\ (\mathbf{L}_\infty(x_3)u, u)_{\mathbb{C}^N} &\le \mu^+ \, \lambda_D(\mathcal{B}^*)^{-1} \, m_*^{-1} \, |u|^2_{\mathbb{C}^N}, \end{aligned} \tag{100}$$

where $\lambda_D(\mathcal{B}^*)$ denotes the lowest eigenvalue for the Dirichlet problem associated to the Laplace operator in the domain D.

(ii) If one makes the stronger assumption (the reader will easily check that (101) implies (99)) that there exists

$$S^* = \mathcal{O}^* \setminus \bigcup_{j=1}^{N} \mathcal{O}_j^* \in \mathcal{C}_N$$

and for each $x_3 \in \mathbb{R}$, a conformal mapping $\mathcal{T}_{c,x_3}$ such that (see figure 3)

$$\mathcal{T}_{c,x_3}(S_{x_3}) \subset \mathcal{S}^*, \tag{101}$$

then there exists a positive constant $A(S^*)$, depending only on S^*, such that (a characterization of $A(S^*)$ is given in the proof below)

$$\begin{aligned} (\mathbf{C}_\infty(x_3)u, u)_{\mathbb{C}^N} &\le \varepsilon^+ \, A(S^*) \, |u|^2_{\mathbb{C}^N}, \\ (\mathbf{L}_\infty(x_3)u, u)_{\mathbb{C}^N} &\ge \mu^- \, A(S^*)^{-1} \, |u|^2_{\mathbb{C}^N}. \end{aligned} \tag{102}$$

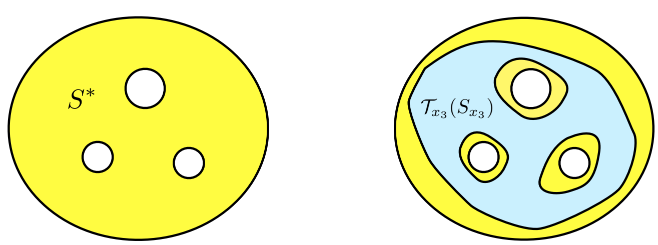


FIGURE 3. Illustration of the assumption (101).

Proof. We simply need to show (100, 102) when $\mathcal{T}_{c,x_3} = \mathbf{I}d$. The proof can then be naturally extended to any conformal mapping using lemma 2.5.

(i) Setting $\varphi(u) := \sum_{j=1}^{N} u_j \; \varphi_{e,j}^\infty(\cdot, x_3)$, we have

$$(\mathbf{C}_\infty(x_3)u, u)_{\mathbb{C}^N} = \int_{S_{x_3}} \varepsilon(\cdot, x_3) \left|\nabla\varphi(u)\right|^2 \; \ge \varepsilon^-(x_3) \int_{S_{x_3}} \left|\nabla\varphi(u)\right|^2.$$

We extend $\varphi(u)$ as a function $\widetilde{\varphi}(u) \in H_0^1(\mathcal{B}^*)$, piecewise constant outside S_{x_3}:

$$\widetilde{\varphi}(u) = \varphi(u) \text{ in } S_{x_3}, \quad \widetilde{\varphi}(u) = 0 \text{ in } \mathcal{B}^* \setminus \mathcal{O}^{x_3}, \quad \widetilde{\varphi}(u) = u_i \text{ in } \mathcal{O}_j^{x_3},\ j = 1, ..., N,$$

in such a way that $\displaystyle\int_{S_{x_3}} \big|\nabla\varphi(u)\big|^2 = \int_{\mathcal{B}^*} \big|\nabla\widetilde{\varphi}(u)\big|^2$. Therefore, since $\bigcup \mathcal{O}_j^{x_3} \subset \mathcal{B}^*$,

$$\begin{aligned}(\mathbf{C}_\infty(x_3)u, u)_{\mathbb{C}^N} &\geq \varepsilon^-(x_3)\ \lambda_D(\mathcal{B}^*)\ \sum_{j=1}^N \int_{\mathcal{O}_j^{x_3}} \big|\widetilde{\varphi}(u)\big|^2 \\ &= \varepsilon^-(x_3)\ \lambda_D(\mathcal{B}^*)\ \sum_{j=1}^N |u_j|^2\ meas(\mathcal{O}_j^{x_3}).\end{aligned}$$

It is then easy to obtain the first inequality of (100). To obtain the second inequality, in the above reasoning, it suffices to replace ε by μ^{-1}, $\varphi_{e,j}^\infty(\cdot, x_3)$ by

$$\varphi_{m,j}^\infty(\cdot, x_3) := \varphi(\mu^{-1}(\cdot, x_3), S_{x_3}) \tag{103}$$

and to use lemma 2.4 that says that $\mathbf{L}_\infty(x_3)^{-1} = \mathbf{Y}(\mu^{-1}(\cdot, x_3), S_{x_3})$. The details are left to the reader. Note that one obtains a lower bound for $\mathbf{L}_\infty(x_3)^{-1}$, which results into an upper bond for $\mathbf{L}_\infty(x_3)$.

(ii) By the Dirichlet principle, we have

$$(\mathbf{C}_\infty(x_3)u, u)_{\mathbb{C}^N} = \inf_{\varphi \in V(x_3, u)} \int_{S_{x_3}} \varepsilon(\cdot, x_3)\, \big|\nabla\varphi\big|^2,$$

where $V(x_3, u) = \big\{\varphi \in H^1(S_{x_3}) \ /\ \varphi|_{\Sigma_j^{x_3}} = u_j,\ \varphi|_{\partial\mathcal{O}^{x_3}} = 0\big\}$. Let us set

$$\varphi_j^* = \varphi_j(S^*, 1) \in H^1(S^*) \text{ and } \varphi^*(u) = \sum_{j=1}^N u_j\, \varphi_j^* \in H^1(S^*).$$

We can extend $\varphi^*(u)$ in S_{x_3} as a function $\widetilde{\varphi}^*(u)$ in $H^1(S_{x_3})$ by

$$\widetilde{\varphi}^*(u) = \varphi^*(u) \text{ in } S^*,\ \widetilde{\varphi}^*(u) = 0 \text{ in } \mathcal{O}^{x_3} \setminus \mathcal{O}^*,\ \widetilde{\varphi}^*(u) = u_i \text{ in } \mathcal{O}_j^{x_3} \setminus \mathcal{O}_j^*,\ 1 \leq j \leq N.$$

By construction $\widetilde{\varphi}^*(u) \in V(x_3, u)$. Thus

$$(\mathbf{C}_\infty(x_3)u, u)_{\mathbb{C}^N} \leq \int_{S_{x_3}} \varepsilon(\cdot, x_3)\, \big|\nabla\widetilde{\varphi}^*(u)\big|^2 = \int_{S^*} \varepsilon(\cdot, x_3)\, \big|\nabla\varphi^*(u)\big|^2.$$

It is then easy to obtain the first inequality of (102) with

$$A(S^*) = \sup_{u \in \mathbb{C}^N,\ |u|=1} \int_{S^*} \big|\nabla\varphi^*(u)\big|^2.$$

To prove the second inequality, we use again that $\mathbf{L}_\infty(x_3)^{-1} = \mathbf{Y}(\mu^{-1}(\cdot, x_3), S_{x_3})$ and apply the above reasoning to obtain un upper bound for $\mathbf{L}_\infty(x_3)^{-1}$. This leads to the announced result. □

5.3. **Propagation velocities.** In the absence of source, the generalized telegrapher's model (90) is a zero order perturbation of the symmetric hyperbolic system (as it is the case for instance in [4])

$$\left\{\begin{aligned} &\mathbf{C}_\infty(x_3)\ \partial_t\mathbf{V} + \partial_3\,\mathbf{I} = 0, \\ &\mathbf{L}_\infty(x_3)\ \partial_t\,\mathbf{I} + \partial_3\,\mathbf{V} = 0. \end{aligned}\right. \tag{104}$$

In the homogeneous case ($\mathbf{C}_\infty(x_3) = \mathbf{C}_\infty, \mathbf{L}_\infty(x_3) = \mathbf{L}_\infty$), looking for traveling wave solutions of the form

$$\mathbf{V}(x_3,t) = \mathbf{V}_a\, f(x_3 - \lambda t), \quad \mathbf{I}(x_3,t) = \mathbf{I}_a\, f(x_3 - \lambda t), \quad \lambda \in \mathbb{R}$$

leads to characterize the propagation velocities λ via the eigenvalue problem

$$\mathbf{C}_\infty^{-1}\,\mathbf{L}_\infty^{-1}\,\mathbf{I}_a = \lambda^2\,\mathbf{I}_a \quad \Longleftrightarrow \quad \mathbf{C}_\infty^{-1}\,\mathbf{L}_\infty^{-1}\,\mathbf{V}_a = \lambda^2\,\mathbf{V}_a, \quad (\mathbf{V}_a = \mathbf{L}_\infty \mathbf{I}_a).$$

In other words, $\pm\,\lambda$ is an admissible velocity associated to (104) if and only if

$$\lambda^2 \in \sigma\big(\mathbf{C}_\infty^{-1}\,\mathbf{L}_\infty^{-1}\big),$$

where $\sigma\big(\mathbf{C}_\infty^{-1}\,\mathbf{L}_\infty^{-1}\big) \subset \mathbb{R}_*^+$ since $\mathbf{C}_\infty$ and $\mathbf{L}_\infty$ are real symmetric positive definite.

By extension, the (variable in space) propagation velocities associated to the system (104) at point x_3 are the reals $\pm\,\lambda(x_3)$ such that

$$\lambda(x_3)^2 \in \sigma\big(\mathbf{C}_\infty(x_3)\,\mathbf{L}_\infty(x_3)\big). \tag{105}$$

Our goal is to provide lower and upper bounds for these velocities. Not surprisingly, these bounds will be related to the function

$$c(x,x_3) = \big(\varepsilon(x,x_3)\,\mu(x,x_3)\big)^{-\frac{1}{2}}, \tag{106}$$

which represents the velocity of electromagnetic waves at point (x,x_3). The first result in this direction is:

Lemma 5.4. *Assume that the velocity of electromagnetic waves is constant in the cross section S_{x_3}:*

$$c(x,x_3) = c(x_3) \tag{107}$$

then

$$\mathbf{C}_\infty^{-1}(x_3)\,\mathbf{L}_\infty^{-1}(x_3) = c(x_3)^2\,\mathbf{I}d.$$

Proof. Let us remind that

$$\mathbf{C}_\infty(x_3) = \mathbf{Y}\big(\varepsilon(\cdot,x_3), S_{x_3}\big) = c(x_3)^{-2}\,\mathbf{Y}\big(\mu(\cdot,x_3)^{-1}, S_{x_3}\big),$$

where we have used (106) and (107). According to lemma 2.4,

$$\mathbf{C}_\infty(x_3) = c(x_3)^{-2}\,\mathbf{Z}\big(\mu(\cdot,x_3), S_{x_3}\big)^{-1} = c(x_3)^{-2}\,\mathbf{L}_\infty(x_3)^{-1}.$$

□

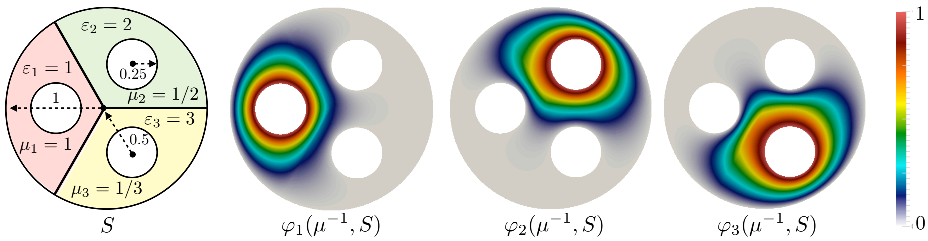
The above result is not true in general as it can be illustrated by the following numerical results. We consider two cross sections with the same geometry and three holes. These two configurations are described in figure 4.

In one case, the function $c(x)$ is constant and on the other case it is not. The results that we obtain via a finite element calculation are the following: for the first situation $c(x) = 1$, and the computed matrices $(\mathbf{C}_\infty, \mathbf{L}_\infty)$ are given by

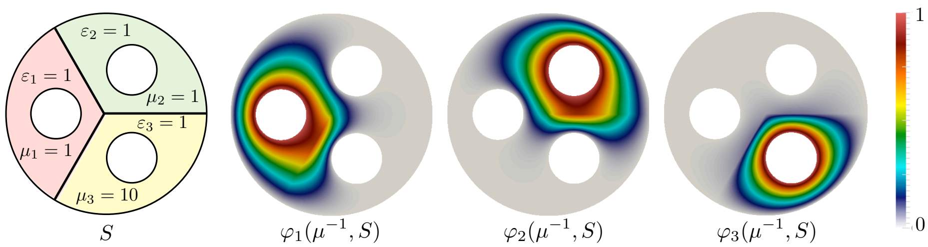
$$\mathbf{C}_\infty = \mathbf{L}_\infty(x_3)^{-1} \simeq \begin{pmatrix} 7.57 & -1.42 & -1.71 \\ -1.42 & 13.2 & -2.91 \\ -1.71 & -2.91 & 17.9 \end{pmatrix},$$

whereas in the second case, $c(x) \in \{1, 1/\sqrt{10}\}$ and we find

$$\mathbf{C}_\infty \simeq \begin{pmatrix} 6.72 & -1.14 & -1.14 \\ -1.14 & 6.72 & -1.14 \\ -1.14 & -1.14 & 6.72 \end{pmatrix}, \quad \mathbf{L}_\infty^{-1} \simeq \begin{pmatrix} 5.89 & -1.25 & -0.19 \\ -1.25 & 5.89 & -0.19 \\ -0.19 & -0.19 & 0.83 \end{pmatrix}.$$



(A) First test case: $c(x) = 1$.



(B) Second test case: $c(x)$ non-constant.

FIGURE 4. Left: the two different configurations. Right: level lines of the functions $\varphi_j(\mu^{-1}, S)$.

A generalization of the result of lemma 5.4 is obtained via lower and upper bounds for the propagation velocities:

Lemma 5.5. *Setting* $c^-(x_3) = \inf_{x \in S_{x_3}} c(x, x_3)$ *and* $c^+(x_3) = \sup_{x \in S_{x_3}} c(x, x_3)$, *we have:*

$$\lambda(x_3)^2 \in \sigma\big(\mathbf{C}_\infty^{-1}(x_3)\,\mathbf{L}_\infty^{-1}(x_3)\big) \quad \Longrightarrow \quad c^-(x_3) \leq |\lambda(x_3)| \leq c^+(x_3) \tag{108}$$

Proof. Let $(u, v) \in \mathbb{C}^N \times \mathbb{C}^N$, $\varphi(u) := \sum_{j=1}^N u_j \; \varphi_{e,j}^\infty(\cdot, x_3)$ and $\psi(v) := \sum_{j=1}^N v_j \; \psi_{m,j}^\infty(\cdot, x_3)$.

Using the biorthogonality formulas (21) with $\rho_1 = \varepsilon(\cdot, x_3), \rho_2 = \mu(\cdot, x_3)$, we get

$$(u, v)_{\mathbb{C}^N} = \int_{S_{x_3}} \mathbf{rot}\,\varphi(u) \cdot \overline{\widetilde{\nabla}\psi(v)}.$$

Then, since $c(\cdot, x_3)\, \varepsilon(\cdot, x_3)^{\frac12} \mu(\cdot, x_3)^{\frac12} = 1$, by definition (106) of $c(\cdot, x_3)$,

$$\begin{aligned} \big|\,(u, v)_{\mathbb{C}^N}\,\big| \quad &\leq \quad \int_{S_{x_3}} c(\cdot, x_3)\; \varepsilon(\cdot, x_3)^{\frac12} \big|\nabla\varphi(u)\big| \; \mu(\cdot, x_3)^{\frac12} \big|\widetilde{\nabla}\psi(v)\big| \\ &\leq \quad c^+(x_3) \,\Big(\int_{S_{x_3}} \varepsilon(\cdot, x_3)\,\big|\nabla\varphi(u)\big|^2\Big)^{\frac12} \Big(\int_{S_{x_3}} \mu(\cdot, x_3)\,\big|\widetilde{\nabla}\psi(v)\big|^2\Big)^{\frac12}, \end{aligned}$$

which can be rewritten, using the definition of $\mathbf{C}_\infty(x_3)$ and $\mathbf{L}_\infty(x_3)$,

$$\big|\,(u, v)_{\mathbb{C}^N}\,\big| \leq c^+(x_3)\, \big|\,(\mathbf{C}_\infty(x_3)u, u)_{\mathbb{C}^N}\,\big|^{\frac12} \; \big|\,(\mathbf{L}_\infty(x_3)v, v)_{\mathbb{C}^N}\,\big|^{\frac12}. \tag{109}$$

It is easy to see that saying $\lambda(x_3)^2 \in \sigma\big(\mathbf{C}_\infty(x_3)^{-1}\,\mathbf{L}_\infty(x_3)^{-1}\big)$ is equivalent to

$$|\lambda(x_3)| \in \sigma\big(\mathbf{C}_\infty(x_3)^{-\frac{1}{2}}\,\mathbf{L}_\infty(x_3)^{-\frac{1}{2}}\big),$$

thus, there exists $w \in \mathbb{C}^N$, with $|w|_{\mathbb{C}^N} = 1$, such that

$$\mathbf{L}_\infty(x_3)^{-\frac{1}{2}}\; w = |\lambda(x_3)|\; \mathbf{C}_\infty(x_3)^{\frac{1}{2}}\; w.$$

Therefore, choosing $u = \mathbf{C}_\infty(x_3)^{-\frac{1}{2}}\; w$ and $v = \mathbf{L}_\infty(x_3)^{-\frac{1}{2}}\; w$, we have

$$(u, v)_{\mathbb{C}^N} = \big(\mathbf{C}_\infty(x_3)^{-\frac{1}{2}}\; w, \mathbf{L}_\infty(x_3)^{-\frac{1}{2}}\; w\big)_{\mathbb{C}^N} = |\lambda(x_3)|,$$

while

$$(\mathbf{C}_\infty(x_3)u, u)_{\mathbb{C}^N} = |w|^2_{\mathbb{C}^N} = 1 \text{ and } (\mathbf{L}_\infty(x_3)v, v)_{\mathbb{C}^N} = |w|^2_{\mathbb{C}^N} = 1.$$

Then, (109) leads to $|\lambda(x_3)| \leq c^+(x_3)$.

For the lower bound, we repeat the same reasoning as above with the functions defined by (103), $\psi^\infty_{e,j}(\cdot, x_3) := \psi(\varepsilon^{-1}(\cdot, x_3), S_{x_3})$ and

$$\varphi(u) := \sum_{j=1}^{N} u_j\; \varphi^\infty_{m,j}(\cdot, x_3) \text{ and } \psi(v) := \sum_{j=1}^{N} v_j\; \psi^\infty_{e,j}(\cdot, x_3).$$

We then get

$$\big|\,(u, v)_{\mathbb{C}^N}\,\big| \leq c^-(x_3)^{-1}\,\Big(\int_{S_{x_3}} \varepsilon(\cdot, x_3)^{-1}\,\big|\nabla\varphi(u)\big|^2\Big)^{\frac{1}{2}}\Big(\int_{S_{x_3}} \mu(\cdot, x_3)^{-1}\,\big|\widetilde{\nabla}\psi(v)\big|^2\Big)^{\frac{1}{2}}$$

that is to say, instead of (109),

$$\big|\,(u, v)_{\mathbb{C}^N}\,\big| \leq c^-(x_3)^{-1}\,\big|\,(\mathbf{C}_\infty(x_3)^{-1}\,u, u)_{\mathbb{C}^N}\,\big|^{\frac{1}{2}}\;\big|\,(\mathbf{L}_\infty(x_3)^{-1}\,v, v)_{\mathbb{C}^N}\,\big|^{\frac{1}{2}}. \tag{110}$$

Then, choosing $u = \mathbf{C}_\infty(x_3)^{\frac{1}{2}}\; w$ and $v = \mathbf{L}_\infty(x_3)^{\frac{1}{2}}\; w$, leads to $|\lambda(x_3)| \geq c^-(x_3)$.

$\square$

5.4. **A remark on undetectable defects.**

We finish this section by a remark linked to the use of our models in the context of non destructive testing.

First, let us consider two reference cables

$$(\Omega, \varepsilon, \mu, \sigma_e, \sigma_m) \quad \text{and} \quad (\widetilde{\Omega}, \widetilde{\varepsilon}, \widetilde{\mu}, \widetilde{\sigma}_e, \widetilde{\sigma}_m),$$

such that for any $x_3 \in \mathbb{R}$, there exists a conformal mapping $\mathcal{T}_{c,x_3}$ (cf. (31)) such that

$$\forall\; x_3 \in \mathbb{R}, \quad \widetilde{S}_{x_3} = \mathcal{T}_{c,x_3}(S_{x_3}) \tag{111}$$

as well as

$$\widetilde{\varepsilon}(\cdot, x_3) = \varepsilon(\cdot, x_3) \circ \mathcal{T}_{c,x_3}, \qquad \widetilde{\mu}(\cdot, x_3) = \mu(\cdot, x_3) \circ \mathcal{T}_{c,x_3},$$

$$\widetilde{\sigma}_e(\cdot, x_3) = \sigma_e(\cdot, x_3) \circ \mathcal{T}_{c,x_3}, \qquad \widetilde{\sigma}_m(\cdot, x_3) = \sigma_m(\cdot, x_3) \circ \mathcal{T}_{c,x_3}.$$

Then as a consequence of lemma 2.5, the two effective 1D telegrapher's models associated to the two reference cables are the same.

In the context of non destructive testing, let us first consider an ideal perfectly cylindrical cable corresponding to:

$$\Omega^* = S \times \mathbb{R}$$

and coefficients $(\varepsilon^*, \mu^*, \sigma_e^*, \sigma_m^*)$ that are only functions of the transverse variable x. Let us consider a cable with a local defect defined, according to our example, by

$$S_{x_3} = \mathcal{T}_{c,x_3}(S),$$

where each $\mathcal{T}_{c,x_3}$ is a diffeomorphism of $\mathbb{R}^2$ such that

$$\mathcal{T}_{c,x_3} \equiv \mathbf{I}d \ \text{ for } \ x_3 \notin [a,b], \quad \mathcal{T}_{c,x_3} \neq \mathbf{I}d \ \text{ for } \ x_3 \in [a,b],$$

which corresponds to a geometrical defect localized in the portion $[a,b]$ of the cable. Assume that the coefficients for this cable are given by

$$\varepsilon(\cdot, x_3) = \varepsilon^* \circ \mathcal{T}_{c,x_3}, \qquad \mu(\cdot, x_3) = \mu^* \circ \mathcal{T}_{c,x_3},$$

$$\sigma_e(\cdot, x_3) = \sigma_e^* \circ \mathcal{T}_{c,x_3}, \qquad \sigma_m(\cdot, x_3) = \sigma_m^* \circ \mathcal{T}_{c,x_3}.$$

Then, if one assumes that, for $x_3 \in [a,b]$, $\mathcal{T}_{c,x_3}$ is a conformal mapping (cf. (31)), then it is not possible to detect the defect with the use of electromagnetic waves by simply using our effective model.

In such a situation, either the original 3D model, or hopefully a higher order asymptotic model, is needed.

An example: the Joukowsky's transformation. To illustrate our discussion we consider the specific case of two homogeneous straight coaxial cables whose sections are obtained one form the other using the Joukoswky's conformal mapping:

$$\mathcal{J}(x_1,x_2) = \big(\, x_1 + x_1/(x_1^2+x_2^2)\,,\, x_2 - x_2/(x_1^2+x_2^2)\,\big)^T,$$

which, in particular maps circles into ellipses. In the numerical results represented in Figure 5, the reference sections S considered are annuli of inner radius r_- and outer radius r_+. In that specific case (one hole, homogenous straight cables) the (scalar) capacitances for the section S and $\mathcal{J}(S)$ are the same and are given by

$$\mathbf{C}_\infty = \varepsilon \parallel \nabla\varphi_1(1,S) \parallel_S^2 = \varepsilon \parallel \nabla\varphi_1\big(1,\mathcal{J}(S)\big) \parallel_S^2.$$

Figure 5 we plot the annulus and its transform by $\mathcal{J}$ in the case $(r_-, r_+) = (1.2, 2)$.

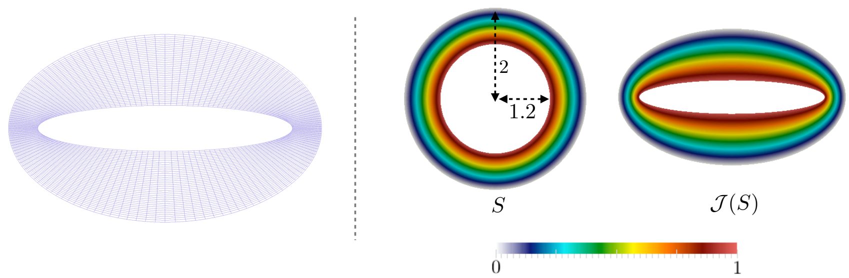


FIGURE 5. Left: Computational mesh. Right: level lines of $\varphi_1\big(1,S\big)$ and $\varphi_1\big(1,\mathcal{J}(S)\big)$.

For the annulus it is is know that the (scalar) capacitance is given by

$$\mathbf{C}_\infty = \frac{2\pi\varepsilon}{\ln(r_+/r_-)} \simeq 12.30005899\,\varepsilon.$$

For the transformed domain $\mathcal{J}(S)$, we have used a $Q5$-finite element method and the computational mesh Figure 5 to compute the capacitance numerically, which gave $\mathbf{C}_\infty \simeq 12.3001\,\varepsilon$.

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